Symbolic and quantitative representations of uncertainty: an overview

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July 2017

ECSQARU/ISIPTA 2017, Lugano, CH

Introduction

Uncertainty is present when an agent cannot assign a truth-value to a proposition, and can just express (degrees of) belief that this proposition is true or not

Uncertainty modeling often appears in two forms

Aleatory uncertainty

- originated from the representation of random phenomena.
- naturally measured by (limit of) frequencies of occurrence of events in repeated experiments
- probability theory is used, where

$$\text{Belief(future event)} = \text{Frequency(past occurrences)}$$

- has some objective flavor

Epistemic uncertainty

- originated from the lack of information of agents.
- more naturally qualitative than quantitative : uses sets of possible values.
- many proposals to represent beliefs : from subjective (imprecise) probability to logic
- is essentially subjective (agent-dependent)

- Difficult to find one's way in the variety of proposals....
- Can have epistemic uncertainty about frequencies, and repeated imprecise observations.

Why not just probabilities ?

Subjective probability is the favorite approach to represent belief

- Assessment methods clear for the degree of belief $b(A)$
  - Analogical method : compared to the frequency of the event of drawing a ball from a known urn.
  - Betting approach : fair price of a bet on the occurrence of A
- Easy to compute with, good properties
- Rooted in a long respectable tradition
- Mainstream

BUT

Single probabilities cannot represent ignorance

- Boolean probability = deterministic precise information
- Uniform probability (ignorance) is not stable via rescaling
- Cannot assign the same degree of belief to all contingent events

Information items

The basic block for representing information is the information item denoted by T that provides information about some entity x valued on some frame of discernment Ω

Essential characteristics :

Support : A non-empty set $\mathcal{S}(T) \subseteq \Omega$

that contains the set of values considered not impossible by information T.

Core A set $\mathcal{C}(T) \subseteq \Omega$

that contains the set of most plausible values according to T .

Plausibility ordering on Ω defined by T

a partial preorder $\succeq_T$: $\omega_1 \succeq_T \omega_2$ means that ω_1 is at least as plausible as ω_2 according to T.

Information items

$T \mapsto (\mathcal{C}(T), \mathcal{S}(T), \succeq_T)$

- $\forall \omega_1 \in \mathcal{S}(T), \omega_2 \notin \mathcal{S}(T)$ implies $\omega_1 \succ_T \omega_2$.
- the core is made of the maximal elements of $\succeq_T$

Extreme cases

- *Total ignorance* : $T^\top$ such that $\mathcal{C}(T) = \Omega$.
  It represents vacuous information $(\omega \sim_T \omega', \forall \omega, \omega' \in \Omega)$.
- *Complete knowledge* : T^ω such that $\mathcal{S}(T^\omega) = \{\omega\}$
  (the actual world is known).

Various representations of information items

An information item can be encoded in various settings :

- Boolean framework
  - *epistemic sets* : Just a support $\mathcal{S}(T)$ excluding impossible values
  - *Logic* : Propositional logic, epistemic modal logics
- Qualitative framework
  - Plausibility orderings on possible worlds or on events
  - Comparative probabilities, etc.
  - Qualitative possibility distributions valued on a finite scale L
  - Many-valued logics
- Quantitative frameworks using set-functions
  - Possibility distributions : fuzzy sets
  - Probability distributions(but modeling ignorance is problematic)
  - Belief functions (weighted epistemic sets)
  - Credal sets (convex sets of probabilities)

Aim of the talk

- What is common to set-functions, modalities and truth-tables ?
- Similarities between qualitative and quantitative approaches
  Sophisticated numerical approaches often have coarse qualitative counterparts
- Information coming from the merging of unreliable testimonies : handling contradictions

Claim

Capacities as the unifying concept for handling incomplete and inconsistent information

Outline

1. Modal logic, three-valued logics and possibility theory
2. Logics of graded belief
3. Qualitative vs. quantitative capacities : analogies
4. Handling multisource information

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Reasoning about beliefs (and uncertainty) : traditions

Three main traditions

- *The probabilistic tradition* : subjective probability (De Finetti, Ramsey), possibility theory, belief function, imprecise probability using set-functions
- *The multiple-valued logic tradition* (Łukasiewicz, Kleene, Belnap) : reasoning with incomplete and contradictory information using truth-tables
- *The modal logic tradition* : (von Wright, Hintikka, Halpern...) epistemic and doxastic modal logics, based on Kripke accessibility relations

Possibility theory is instrumental to bridge these approaches.

Boolean approach to incomplete information : 2-valued possibility theory

Based on an epistemic state described by a set $\emptyset \neq E \subseteq \Omega$, we can define possibility and necessity degrees $N(A)$ and $\Pi(A)$ by

- $\Pi(A) = 1$ if and only if $E \cap A \neq \emptyset$ and 0 otherwise
- $N(A) = 1$ if and only if $E \subseteq A$ and 0 otherwise.

N is called a necessity measure and Π a possibility measure.

- $N(A) = 1$ means that A is certainly true, and $\Pi(A) = 0$ that A is certainly false,
- In particular, if $N(A) = 0$ and $\Pi(A) = 1$ it means that the truth of A is unknown in epistemic state E.

Modal Epistemic logic and possibility theory

This framework fits the one of modal logic, where $\Box\phi$ means $N(\phi) = 1$

	Boolean possibility theory	KD Modal logic
Tools	set functions N, Π	modalities $\Box, \Diamond$
Scale	$\{0, 1\}$	$\{0, 1\}$
Adjunction	$N(\phi \wedge \psi) = \min(N(\phi), N(\psi))$	$\Box(\phi \wedge \psi) \equiv \Box\phi \wedge \Box\psi$
Duality	$\Pi(\phi) = 1 - N(\neg\phi)$	$\Box\phi \equiv \neg\Diamond\neg\phi$
Axiom D	$\Pi(\phi) \geq N(\phi)$	$\Box\phi \to \Diamond\phi$

but if N is a probability then E is a singleton !

A minimal two-tiered epistemic logic (MEL)

(Banerjee, Dubois, 2009)

1. Standard propositional Boolean logic language $\mathcal{L}$
   - Ontic propositional variables $\mathcal{V} = \{a, b, c, \ldots, p, \ldots\}$
   - $\alpha, \beta, \ldots$ propositional formulae of $\mathcal{L}$ built using conjunction, disjunction, and negation $(\wedge, \vee, \neg)$
2. Modal level : A propositional language $\mathcal{L}_\Box$
   - Epistemic propositional variables : $\mathcal{V}_\Box = \{\Box\alpha : \alpha \in \mathcal{L}\}$
   - $\mathcal{L}_\Box$ propositional language based on $\mathcal{V}_\Box$

MEL is the minimal language to express partial knowledge about the truth of propositions. (you can write "the agent ignores α" as $\neg\Box\alpha \wedge \neg\Box\neg\alpha$)
$\Rightarrow$ The "subjective" fragment of KD (or S5) without modality nesting.

The MEL axioms

MEL is based on the language $\mathcal{L}_\Box$ with the propositional axioms

(PL) Axioms of PL for $\mathcal{L}_\Box$-formulas

(K) $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$

(D) $\Box\alpha \to \Diamond\alpha$

(Nec) $\Box\alpha$, for each $\alpha \in \mathcal{L}$ that is a PL tautology, i.e. if $Mod(\alpha) = \Omega$.

the inference rule is modus ponens.

It is a two-tiered propositional logic, not a full-fledged modal logic :

$$\mathcal{B} \vdash_{MEL} \Phi \iff \mathcal{B} \cup \{K, D, Nec\} \vdash_{PL} \Phi$$

Possibilistic semantics

The semantics does not require accessibility relations

- A classical interpretation of MEL is equivalent to an epistemic state, a non-empty set $E \subseteq \Omega$ of Boolean interpretations, or equivalently a Boolean necessity measure.

$$t(\Box\alpha) = 1 \iff N_t(\alpha) = 1 \iff E_t \subseteq [\alpha] \iff E_t \models \Box\alpha$$

Satisfiability

$E \models \Box\alpha$ means that α is true in all worlds compatible with the epistemic state E (as usual in epistemic logic)

The models of $\Box\alpha$ are $\{E \neq \emptyset : E \subseteq [\alpha]\} \subseteq 2^{\Omega}$

MEL is sound and complete with respect to this semantics

Is MEL a modal logic in the usual sense ?

	Doxastic logic KD45	MEL
Syntax	Nested modal formulas	No nesting
Semantics	Accessibility relations	Sets and induced set-functions
Axioms	S4, S5	None
Scope	Introspection	Information from an agent

Previous related works relating uncertainty and modal logic

- Petruszczak (2009) showed that Kripke semantics of K45 and similar logics can be drastically simplified.
- Existence of various approaches to relate probability and modal logics where $\Box\alpha$ stands for $P([\alpha]) \geq \lambda$.
  Hamblin (1959), Burgess (1969), Walley and Fine (1979), Logic of Risky Knowledge (Kyburg and Teng, 2002)
- Smets (1988) noticed that KD45 modalities are related to Shafer belief functions : $Bel(\alpha) = P(\Box\alpha)$.
- *Fuzzy logic approach* (Hajek, Godo...) : $\Box\alpha$ is many-valued : the truth-value of a modal proposition is the degree of belief in the proposition

Kleene logic : a logic of incomplete information

Truth values $\mathbb{T}_3 = \{\mathbf{0} < \frac{\mathbf{1}}{\mathbf{2}} < \mathbf{1}\}$, where $\frac{\mathbf{1}}{\mathbf{2}}$ is a third truth value referring to *unknown*.

- *Syntax* : the same connectives as classical logic : $(\wedge, \vee, \neg)$
- *Negation* : $\mathbf{t}(\neg\alpha) = n(\mathbf{t}(\alpha))$, where $n(\mathbf{1}) = \mathbf{0}$, $n(\mathbf{0}) = \mathbf{1}$ and $n(\frac{\mathbf{1}}{\mathbf{2}}) = \frac{\mathbf{1}}{\mathbf{2}}$;
- *Conjunction* : $\mathbf{t}(\alpha \wedge \beta) = \min(\mathbf{t}(\alpha), \mathbf{t}(\beta))$;
- *Disjunction* : $\mathbf{t}(\alpha \vee \beta) = \max(\mathbf{t}(\alpha), \mathbf{t}(\beta))$, by De Morgan laws
- *Implication* : $p \to_K q \equiv \neg p \vee q$.

Paradox ? There are no tautologies, in particular
$\mathbf{t}(\alpha \wedge \neg\alpha) = \mathbf{t}(\alpha \vee \neg\alpha) = \frac{\mathbf{1}}{\mathbf{2}}$ when $\mathbf{t}(\alpha) = \frac{\mathbf{1}}{\mathbf{2}}$

"Unknown" is not a truth value in the usual ontic sense

- In practice, $\frac{1}{2}$ is used to *model the idea that the truth-value of a Boolean proposition is unknown*.
- "'Unknown" is in conflict with "Known to be true" and "Known to be false, not with "true" and "false".
- One must distinguish between two levels :
  - **Ontic values** : true (T), false (F)
  - **Epistemic values are possibility distributions on** $\{F, T\}$: certainly true $\mathbf{1} = \{T\}$, certainly false $\mathbf{0} = \{F\}$, $\frac{1}{2} = \{F, T\}$
- The three-valued $\mathbf{t}(\alpha)$ encodes knowledge about the truth $t(\alpha)$ of a Boolean proposition α.

Translating 3-valued propositional atoms

(with D. Ciucci)
$\mathcal{T}(\mathbf{t}(a) \in T)$: translation into MEL of the assertion $\mathbf{t}(a) \in T \subseteq \mathbf{3}$

$$\mathcal{T}(\mathbf{t}(a) = \mathbf{1}) = \Box a \text{ (certainty of } a)$$
$$\mathcal{T}(\mathbf{t}(a) = \mathbf{0}) = \Box \neg a$$
$$\mathcal{T}(\mathbf{t}(a) = \tfrac{1}{2}) = \Diamond a \wedge \Diamond \neg a = \neg \Box a \wedge \neg \Box \neg a \text{ (ignorance)}$$
$$\mathcal{T}(\mathbf{t}(a) \geq \tfrac{1}{2}) = \Diamond a \text{ (possibility of } a)$$
$$\mathcal{T}(\mathbf{t}(a) \leq \tfrac{1}{2}) = \Diamond \neg a$$

Formally, "$\mathbf{t}(a) \in T$" stands for $\{\mathbf{t} : \mathbf{t}(a) \in T\}$ so the translation is from $2^{\mathbf{V}_3}$ to $\mathcal{L}_{\Box}$.

Translation of Kleene logic into MEL

- Elementary connectives on atoms :
  - Conjunction : $\mathcal{T}(\mathbf{t}(a \wedge b) = \mathbf{1}) = \Box a \wedge \Box b$
  - Disjunction : $\mathcal{T}(\mathbf{t}(a \vee b) = \mathbf{1}) = \Box a \vee \Box b$
  - Negation : $\mathcal{T}(\mathbf{t}(\neg a) = \mathbf{1}) = \Box \neg a$
  - Kleene Implication :$\mathcal{T}(\mathbf{t}(\neg a \vee b) = \mathbf{1}) = \Box \neg a \vee \Box b = \Diamond a \to \Box b$
- A knowledge base B in Kleene logic :
  The translation $\mathcal{T}(B) = \{\mathcal{T}(\mathbf{t}(\beta) = 1) : \beta \in B\}$ in MEL consists in the same conjunction of clauses as B, with modality $\Box$ in front of literals.

$$\mathcal{T}(\{\neg a \vee b, c \vee \neg c\}) = \{\Box \neg a \vee \Box b, \Box c \vee \Box \neg c\}$$

A theorem-preserving translation : $B \vdash_{Kleene} \alpha \iff \mathcal{T}(B) \vdash_{MEL} \mathcal{T}(\alpha)$

The Kleene fragment of MEL

- The Kleene fragment of MEL : $\Box$ in front of literals.

$$\mathcal{L}^K_\Box := \Box a | \Box \neg a | \alpha \vee \beta | \alpha \wedge \beta \subset \mathcal{L}^\ell_\Box$$

- In this fragment, formulas of the form $\neg \Box a$ are not allowed.

We reconcile Boolean logic and the lack of tautologies in Kleene logic

- a is always true or false
- $\Box a \vee \Box \neg a$ is not a tautology
- $\Diamond a \wedge \Diamond \neg a$ is not a contradiction
- This translation into MEL shows the meaning of Kleene implication.

Kleene valuations as partial Boolean models

- Three-valued interpretations $\mathbf{t}$ can be mapped to partial Boolean models where only some literals are known.

$$E_{\mathbf{t}} = [(\wedge_{\mathbf{t}(a)=\mathbf{1}} \; a) \bigwedge (\wedge_{\mathbf{t}(b)=\mathbf{0}} \; \neg b)]$$

- Partial models : one can only express information on each variable independently of other ones.
  (Cannot express information on disjunction of literals.)
- Kleene logic is to MEL and possibility theory what reasoning with product of marginal probabilities is to Bayes nets.

Kleene logic, more generally all three-valued logics, is possibilistic reasoning with a very limited expressive power

Outline

Graded possibility theory

At the bridge between numerical and symbolic representations of uncertainty : expressing that some state of affairs are more plausible than other ones.

A possibility distribution can be :

- **Ordinal** : a complete preorder on Ω : $w \geq_\pi w'$.

Basic properties in the ordinal case

- $A \geq_\Pi B \iff \forall w \in B, \exists w' \in A, w' \geq_\pi w$ and
  $A \geq_N B \iff \overline{B} \geq_\Pi \overline{A}$
- Axioms
  - $A \geq_\Pi B \Rightarrow A \cup C \geq_\Pi B \cup C$ (comparative possibility, Lewis)
  - $A \geq_N B \Rightarrow A \cap C \geq_N B \cap C$ (epistemic entrenchment, Gärdenfors)

Possibility theory

At the bridge between numerical and symbolic representations of uncertainty : expressing that some state of affairs are more plausible than other ones.

A possibility distribution can be :

- **Qualitative** : a mapping $\pi : \Omega \to (L, \min, \max, 1 - \cdot)$

Basic properties in the qualitative case

- $\Pi(A) = \max_{w \in A} \pi(x)$; $N(A) = 1 - \Pi(\overline{A})$
- Axioms : $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$; $N(A \cap B) = \min(N(A), N(B))$
- Characterized by the ordinal axioms.
- can express absolute notions of full possibility and impossibility.

Possibility theory

At the bridge between numerical and symbolic representations of uncertainty : expressing that some state of affairs are more plausible than other ones.

A possibility distribution can be :

- **Quantitative** : a mapping $\pi : \Omega \to [0, 1]$
  - A membership function pertaining to a quantitative linguistic category : "John is tall" $\mapsto \pi = \mu_{Tall}$.
  - A nested family of set E_i with $P(E_i) \geq a_i$ (probabilistic inequalities, confidence intervals)
  - A nested random set (contour function of a consonant belief function)
  - A likelihood function $P(B|x)$ since $P(B|A) \leq \max_{x \in B} P(B|x)$
  - a Spohn kappa function $\kappa : \Omega \to \mathbb{N}$ letting $\pi(w) = 2^{-\kappa(w)}$

Generalized possibilistic logic (D, Prade, Schockaert)

GPL is based on the language $\mathcal{L}_\Box^L$ with atoms $\Box_\lambda \alpha, \alpha \in \mathcal{L}, \lambda \in L \setminus \{0\}$

(PL) Axioms of PL for $\mathcal{L}_\Box$-formulas

(K) $\Box_\lambda(\alpha \to \beta) \to (\Box_\lambda \alpha \to \Box_\lambda \beta)$

(D) $\Box_\lambda \alpha \to \Diamond_1 \alpha$

(Nec) $\Box_\lambda \alpha$, for each $\alpha \in \mathcal{L}$ that is a PL tautology, i.e. if $Mod(\alpha) = \Omega$.

(W) $\Box_\lambda \alpha \to \Box_\mu \alpha$ if $\lambda \geq \mu$ (weakening)

the inference rule is modus ponens.

- Same axioms as MEL for each λ + Weakening
- $\pi \models \Box_\lambda \alpha \iff N([\alpha]) \geq \lambda$
- **Possibilistic logic** : $\Box_\lambda \alpha \equiv (\alpha, \lambda)$, only conjunctions.
- $\Box_\lambda \alpha \wedge \beta \equiv \Box_\lambda \alpha \wedge \Box_\lambda \beta$

Graded belief

- To each proposition A (subset of Ω), assign a confidence degree $\gamma(A) \in L$ (a totally ordered scale)

A q-capacity (or fuzzy measure) is a mapping $\gamma : 2^{\Omega} \to L$ such that

$\gamma(\emptyset) = 0; \gamma(S) = 1$; and if $A \subseteq B$ then $\gamma(A) \leq \gamma(B)$.

Suppose one interprets belief in A ($\Box_{\lambda}\alpha$ with A $= [\alpha]$) as $\gamma(A) \geq \lambda$ for a sufficiently high confidence threshold. To extend the simplified epistemic logic setting to graded belief, one needs the following axiom

Adjunction : If $\gamma(\alpha) \geq \lambda$ and $\gamma(\beta) \geq \lambda$ then $\gamma(\alpha \wedge \beta) \geq \lambda, \forall \lambda \in L$

Then $\gamma(\alpha \wedge \beta) = \min(\gamma(\alpha), \gamma(\beta))$, so that γ is a necessity measure.

Logical accounts of graded uncertainty

Three ways of reasoning based on non-Boolean capacities

- **Graded modalities** : $\Box_\lambda p$ stands for $\gamma([p]) \geq \lambda \in L$
  Probabilistic logic (Nilsson, Halpern...), GPL, logic of risky knowledge.

  No adjunction axiom : non-regular modal logics.

- **Fuzzy logic approach**, e.g. Łukasiewicz logic

  $\Box p$ is many-valued : truth-degree($\Box\alpha$) = belief-degree(α)

  Hajek et. al. (from 1995 on) for possibility, probability, belief functions.

- **Comparative/ conditional approaches**

  Atomic formula $p > q$ stands for $g([p]) > g([q])$

  Lewis comparative possibility, Halpern (1997) on partial order...

Outline

Analogies between quantitative and qualitative capacities

A capacity (or fuzzy measure) is a mapping $\gamma : 2^\Omega \to [0, 1]$ such that

$g(\emptyset) = 0; g(S) = 1$; and if $A \subseteq B$ then $g(A) \leq g(B)$.

Many concepts making sense for numerical capacities (viz. Belief functions, lower probabilities and their duals) have counterparts for qualitative capacities

- Möbius transforms
- Contour functions
- Core (dominating probabilities) and extreme points
- Choquet integrals

However the role played by probability measures in the numerical setting is now played by possibility measures.

Möebius transforms

Numerical	Qualitative
probability distribution $p : \Omega \to [0, 1]$	possibility distribution $\pi : \Omega \to L$
$P(A) = \sum_{w \in A} p(w)$	$\Pi(A) = \max_{w \in A} \pi(w)$
Capacity g	q-capacity γ
Möebius transform	Qualitative Möebius transform
$m_g(A) = \sum_{B \subseteq A} (-1)^{\vert A \setminus B \vert} g(A)$	$\gamma_{\#}(A) = \begin{cases} \gamma(A) \text{ if } > \gamma(A \setminus \{w\}) \\ 0 \text{ otherwise.} \end{cases}$
$g(A) = \sum_{B \subseteq A} m_g(A)$	$\gamma(A) = \max_{B \subseteq A} \gamma_{\#}(B)$
$m_g \geq 0 : g =$ belief function	All γ have positive focals
singleton focals : probability	possibility measures
nested focals : necessity	necessity

Möebius transforms in the qualitative setting

- Focal sets can be viewed as imprecise information from sources (like in belief function theory)
  - Necessity measures : consonant imprecise sources
  - Possibility measures : precise focal sets as conflicting and precise sources
  - Capacities : imprecise and conflicting sources

This view of possibility measure is at odds with the usual view :

- $N(A) = \max_{F \subseteq A} \gamma_{\#}(F)$
  $= \min_{w \notin A} 1 - \pi(w)$ with $\pi(w) = 1 - N(\Omega \setminus w) = \min_{F : w \notin F} 1 - \gamma_{\#}(F)$
  represents imprecise information
- $\Pi(A) = \max_{w \in A} \gamma_{\#}(\{w\})$ where $\pi(w) = \gamma_{\#}(\{w\})$
  represents conflicting information.

Dual vs. upper set-functions

Numerical	Qualitative
Dual capacity $g^c(A) = 1 - g(A^c)$	$\gamma^c(A) = 1 - \gamma(A^c)$
$Pl(A) = \sum_{B \cap A \neq \emptyset} m_g(B) = g^c(A)$	$\gamma^*(A) = \max_{B \cap A \neq \emptyset} \gamma_\#(B)$
$g = Bel \Rightarrow g^c = Pl \geq g$	$\gamma^* \geq \gamma$ but $\gamma^* \neq \gamma^c$, $\gamma^c \not\geq \gamma$
Contour funct. : $Pl(w) = \sum_{w \in B} m_g(B)$	$\pi_\gamma(w) = \max_{w \in B} \gamma_\#(B)$
$Pl(A) \neq \max_{w \in A} Pl(w)$	$\gamma^*(A) = \max_{w \in A} \pi_\gamma(w)$

- In the quantitative case : the dual of a belief function is the expected consistency and is not the possibility measure based on the contour function
- In the qualitative case : the dual of a capacity is not expressing consistency with information sources, but coincides with the possibility measure based on the contour function

Core and qualitative core

Numerical	Qualitative
Core : $C(g) = \{P \geq g\}$	$QC(\gamma) = \{\pi : \Pi \geq \gamma\}$
Sometimes empty	Never empty
Convex set	Sup-semi lattice
Extreme points C^*	minimal elements QC^*
among	among
$\forall i, p_\sigma^g(s_{\sigma(i)}) = g(S_\sigma^i) - g(S_\sigma^{i+1})$	$\forall i, \pi_\sigma^\gamma(s_{\sigma(i)}) = \gamma(S_\sigma^i)$
$g(A) = \min\{P(A) : P \in C^*(g)\}$ if coherence	$\gamma(A) = \min\{\Pi(A) : \pi \in QC^*(\gamma)\}$ always
$g^c(A) = \sup\{P(A) : P \in C(g)\}$	$\gamma(A) = \max\{N(A) : \pi \in QC(\gamma^c)\}$

$S_\sigma^i = \{s_{\sigma(i)}, \ldots, s_{\sigma(n)}\}$ for a permutation σ of the $n = |\Omega|$ elements in S

Outline

Handling inconsistent information

Not only information items are imprecise, but sources of information may conflict

Two approaches

- Merging conflicting information (fusion methods for information items of various nature)
- Reasoning under inconsistency (paraconsistent logics, Belnap setting, argumentation)

Capacities may play a role in both approaches.

General principles for information fusion

Focus : Merging information items T_i, supplied by sources whose reliability levels are not known, yields an information item : $f(T_1, \ldots, T_n)$.

- A symmetric process : the sources play the same role and supply information of the same kind ;
- Information items are considered reliable insofar as it is possible, in order to be useful
- The result should not be arbitrarily precise. It should be faithful to the level of informativeness of the inputs.
- Information fusion should solve conflicts between sources, while neither dismissing nor favoring any of them without a reason.

These principles are embodied in a set of 8 basic postulates.

Information ordering

One must be able to compare information items in terms of relative informativeness.

Information ordering

If T is consistent, $T \sqsubseteq T'$ expresses that T provides at least as much information as T'.

In particular, $T \sqsubseteq T'$ should imply $\mathcal{S}(T) \subseteq \mathcal{S}(T')$ and $\mathcal{C}(T) \subseteq \mathcal{C}(T')$.

- Sets : inclusion.
- Probability : should go beyond comparing entropies, i.e. majorization ordering
- Belief functions, credal sets, confidence relations, capacities : less easy to define $\succeq_T$ (pignistic transform, contour function), $\sqsubseteq$ (specialization, $Bel_1 \leq Bel_2$...)

Fusion of epistemic sets : postulates

Fusion axioms in the set-valued Boolean setting :

- **Unanimity** : Values considered impossible (resp. plausible) by all sources should be impossible (plausible).
- **Monotonicity** : if information items T_i are consistent, and $T_i \sqsubseteq T_i', \forall i = 1, \ldots, n$ then $f(T_1, \ldots, T_n) \sqsubseteq f(T_1', \ldots, T_n')$.
- **Consistency enforcement** : the core of the result should not be empty
- **Optimism** : if information items T_i are consistent, then $f(T_1, \ldots, T_n) \sqsubseteq T_i, \forall i$.
- **Fairness** : $\forall i = 1, \ldots, n, f(T_1, \ldots, T_n)$ is consistent with T_i.
- **Insensitivity to vacuous information** : non-informative sources can be deleted.
- **Minimal commitment** : $f(T_1, \ldots, T_n)$ should be the as little informative as possible while obeying the other postulates.

Fusion of n epistemic sets $T_1, T_2, \ldots T_n$

Optimistic Fairness

For any subset I of consistent sources,

$$f(T_1,\ldots,T_n) \cap \bigcap_{i \in I} T_i \neq \emptyset.$$

- Let $I \subset \{1,..,n\}$ be a maximal consistent subset (MCS) of sources, i.e., $T^I = \cap_i T_{i \in I} \neq \emptyset$ and $T^I \cap \cup_{j \notin I} T_j = \emptyset$.
- MCS({1, . . ., n}) is the set of maximal consistent subsets of sources.

A known combination rule from logic (Rescher and Manor, 1970) is

$$f(T_1, \ldots, T_n) = \bigcup_{I \in MCS(\{1,\ldots,n\})} \bigcap_{i \in I} T_i$$

Optimistic fusion is destructive

Merging set-based information items A and B yields $A \cap B$ or $A \cup B$.

- Original information is lost !
- One cannot retrieve A nor B, if we only have $A \cap B$ or $A \cup B$

Try non destructive fusion using a capacity.

Non-destructive fusion : set-valued case

- Given n information items $A_1, \ldots A_n$, define a basic assignment $\beta : 2^\Omega \to \{0, 1\}$ such that

$$\beta(A_i) = 1, \forall i = 1, \ldots n; \beta(A) = 0 \text{ otherwise.}$$

- Consider the capacity $\gamma(A) = \max_{A_i \subseteq A} \beta(A_i)$.
- From γ, one can recover all non redundant information items A_i, provided that none is less specific than any other : they are minimal sets B such that $\gamma(B) = 1$

Question-answering using Belnap truth-values

- We can describe the epistemic status of each statement $x \in A$ in view of the information provided by the sources :

$\gamma(A), \gamma(\overline{A})$	Interpretation	Belnap truth-value
1 0	$x \in A$ is supported	"true"
0 1	$x \in A$ is negated	"false"
0 0	$x \in A$ is unknown	"unknown"
1 1	$x \in A$ is conflicting	"contradictory"

- These four states form a bilattice for two orderings
  - The truth ordering : 10 $>$ 00 $>$01 ; 10 $>$ 11 $>$01
  - The information ordering : 11 $>$ 01 $>$ 00 ; 11 $>$ 10 $>$ 00

The logic of Boolean capacities BC (D, Prade, Rico)

- The MEL language fragment of the *monotonic modal* logic EMN (no nesting of modalities)

(PL) Axioms of PL for $\mathcal{L}_\Box$-formulas
(RM) : $\Box p \to \Box q$, whenever $\vdash p \to q$.
(N) : $\Box\top$.
(P) : $\Diamond\top$, where $\Diamond p$ stands for $\neg\Box\neg p$.
Modus ponens

- This modal logic is the natural logical account of Boolean capacities
- Semantics : $\gamma \models \Box p$ stands for $\gamma([p]) = 1$
- This logic is sound and complete wrt Boolean capacities.
- No adjunction : $\Box p \wedge \Box q \not\models \Box p \wedge q$
- Its usual semantics is neighbourhood semantics.

Embedding reasoning under inconsistency in the logic of capacities (Ciucci, D, this conf.)

- In BC, $\Box p \equiv \vee_i \Box_i p$, where $\Box_i p$ is a KD modality
- $\gamma \models \Box p \iff (E_1, \ldots, E_n) \models \Box p$ where $E_1, \ldots, E_n$ are focal sets of γ, understood as "p is supported by one source" :

$$E_1 \subseteq [p] \text{ or } \ldots \text{ or } E_n \subseteq [p].$$

- Belnap truth-values and Belnap logic can be captured in this logic.
- If we restrict to capacities such that $\gamma(A \cap B) = \min(\gamma(A), \gamma(B))$ when $A \cap B \neq \emptyset$, then the maximal consistent subset approach is retrieved.

Non-destructive fusion : possibility distributions

Given n information items $\pi_1, \ldots \pi_n$, build the graded capacity

$$\gamma(A) = \max_{i=1}^{n} N_i(A).$$

- From γ, one can recover all information items π_i, provided that none is less specific than any other : they are the most specific elements of the core $\{\pi : N \leq \gamma\}$
- Any capacity is of this form.
- We can describe the epistemic status of each statement $x \in A$ in view of the information provided by the sources using an extension of Belnap bilattice by comparing pairs $(\gamma(A), \gamma(\overline{A}))$

Conclusion

- Capacities as a unifying framework for reasoning under incomplete and conflicting information
- Bridge between symbolic and numerical information : 3 and 4-valued logics, epistemic logic, possibility theory, belief functions, imprecise probabilities
- **Future work** :
  - Link with argumentation : viewing argument supports as sources, and using argument evaluation methods to build capacities
  - Link with imprecise probabilities : beyond coherence

Representing all numerical capacities by conflicting probability sets

$g(A) = \max_{\pi \in QC(g)} \inf_{P \in C(\Pi)} P(A)$ (Brüning and Denneberg)