Exchangeable choice functions

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Options We choose between abstract **options**, collected in a vector space $\mathscr{V}$, ordered by a reflexive vector ordering $\preceq$, whose irreflexive part $\prec$ is defined by $u \prec v \Leftrightarrow (u \preceq v$ and $u \neq v)$ for all u and v in $\mathscr{V}$. $\mathscr{Q}(\mathscr{V})$ is the collection of **non-empty but finite subsets** of $\mathscr{V}$.

Choice functions A **choice function** C on $\mathscr{V}$ is a map

$$C\colon \mathscr{Q}(\mathscr{V}) \to \mathscr{Q}(\mathscr{V}) \cup \{\emptyset\} : A \mapsto C(A) \text{ such that } C(A) \subseteq A.$$

Rationality axioms We call a choice function C on $\mathscr{Q}(\mathscr{V})$ **coherent** if for all A, A_1 and A_2 in $\mathscr{Q}(\mathscr{V})$, all u and v in $\mathscr{V}$, and all λ in $\mathbb{R}_{>0}$:

C1. $C(A) \neq \emptyset$; [avoiding complete rejection]

C2. if $u \prec v$ then $\{v\} = C(\{u,v\})$; [dominance]

C3. a. if $C(A_2) \subseteq A_2 \setminus A_1$ and $A_1 \subseteq A_2 \subseteq A$ then $C(A) \subseteq A \setminus A_1$; [Sen's α]

b. if $C(A_2) \subseteq A_1$ and $A \subseteq A_2 \setminus A_1$ then $C(A_2 \setminus A) \subseteq A_1$; [Aizerman]

C4. a. if $A_1 \subseteq C(A_2)$ then $\lambda A_1 \subseteq C(\lambda A_2)$; [scaling invariance]

b. if $A_1 \subseteq C(A_2)$ then $A_1 + \{u\} \subseteq C(A_2 + \{u\})$. [independence]

To define coherent choice functions, we only need an ordered linear space.

Which uncertainty models do we use? How do choice functions work?

Permutations $\mathscr{P}_n$ is the group of permutations of $\{1,\dots,n\}$. With any π in $\mathscr{P}_n$ and any sequence $X = (X_1,\dots,X_n)$, where each X_i assumes values in the finite set $\mathscr{X}$, we associate its permuted variant

$$\pi(X) = \pi(X_1,\dots,X_n) := (X_{\pi(1)},\dots,X_{\pi(n)}).$$

With any π in $\mathscr{P}_n$ and any gamble f in $\mathscr{L}(\mathscr{X}^n)$, we associate

$$(\pi^t f)(x) := f(\pi(x)) \text{ for all } x \text{ in } \mathscr{X}^n.$$

Exchangeability When the subject assesses a sequence $X = (X_1,\dots,X_n)$ to be **exchangeable**, he is indifferent between any gamble f on $\mathscr{X}^n$ and its permuted variant $\pi^t f$, for any π in $\mathscr{P}$: his set of indifferent gambles is

$$I_{\mathscr{P}_n} := \operatorname{span}\{f - \pi^t f : f \in \mathscr{L}(\mathscr{X}^n), \pi \in \mathscr{P}_n\}.$$

Exchangeability is a special indifference assessment!

Quotient spaces How do we work with this? A set of indifferent options I is **coherent** if for all u and v in $\mathscr{V}$ and λ in $\mathbb{R}$:

I1. $0 \in I$; [indifference to status quo]

I2. if $u \in \mathscr{V}_{\succ 0} \cup \mathscr{V}_{\prec 0}$ then $u \notin I$; [non-triviality]

I3. if $u \in I$ then $\lambda u \in I$; [scaling]

I4. if $u, v \in I$ then $u + v \in I$. [combination]

We collect all options that are indifferent to a given option u in the **equivalence class**

$$[u] := \{v \in \mathscr{V} : v - u \in I\} = \{u\} + I.$$

The set of all these equivalence classes is the **quotient space** $\mathscr{V}/I := \{[u] : u \in \mathscr{V}\}$, which is a vector space with vector ordering

$$[u] \preceq [v] \Leftrightarrow (\exists w \in \mathscr{V})\, u \preceq v + w.$$

for all $[u]$ and $[v]$ in $\mathscr{V}/I$.

Indifference: compatibility with I We call a choice function C on $\mathscr{Q}(\mathscr{V})$ **compatible** with a coherent set of indifferent options I if there is a **representing** choice function C' on $\mathscr{Q}(\mathscr{V}/I)$ such that

$$C(A) = \{u \in A : [u] \in C'(A/I)\}$$

for all A in $\mathscr{Q}(\mathscr{V})$.

Theorem 1. *For any choice function C on $\mathscr{Q}(\mathscr{V})$ that is compatible with some coherent set of indifferent options I, the unique representing choice function C/I on $\mathscr{Q}(\mathscr{V}/I)$ is given by*

$$C/I(A/I) := C(A)/I \text{ for all } A \text{ in } \mathscr{Q}(\mathscr{V}).$$

Moreover, C is coherent if and only C/I is.

Is there a de Finetti-like Representation theorem?

See [*Gert de Cooman & Erik Quaeghebeur, Exchangeability and sets of desirable gambles*] for more information.

Count vectors Since the subject is indifferent between $x = (x_1,\dots,x_n)$ and $(x_{\pi(1)},\dots,x_{\pi(n)})$, a useful statistic of x is its **count vector** $T(x)$, whose z-component $T(x)_z = |\{k \in \{1,\dots,n\} : x_k = z\}|$ for all z in $\mathscr{X}$, and whose range is $\mathscr{N}^n := \{m \in \mathbb{Z}^{\mathscr{X}}_{\geq 0} : \sum_{z\in\mathscr{X}} m_z = n\}$. For any m in $\mathscr{N}^n$, its **invariant atom** is $[m] := \{x \in \mathscr{X}^n : T(x) = m\}$.

A useful map Use the special coherent set of indifferent options $I_{\mathscr{P}_n}$ to find alternative expressions for the equivalent classes $[u] = \{u\} + I_{\mathscr{P}_n}$ and the vector ordering $\preceq$ on $\mathscr{L}(\mathscr{X}^n)/I_{\mathscr{P}_n}$. Consider the special map

$$\mathrm{H}_n \colon \mathscr{L}(\mathscr{X}^n) \to \mathscr{L}(\mathscr{N}^n) : f \mapsto \mathrm{H}_n(f) := \mathrm{H}_n(f|\cdot)$$

where $\mathrm{H}_n(f|m) := \frac{1}{\binom{n}{m}} \sum_{y\in[m]} f(y)$ for all f in $\mathscr{L}(\mathscr{X}^n)$ and m in $\mathscr{N}^n$. H_n is the expectation operator associated with the uniform distribution on $[m]$.

Proposition. *Consider any $[f]$ and $[g]$ in $\mathscr{L}(\mathscr{X}^n)/I_{\mathscr{P}_n}$. Then $[f] \preceq [g] \Leftrightarrow \mathrm{H}_n(f) \leq \mathrm{H}_n(g)$.*

Essentially, H_n is a linear order isomorphism between $\mathscr{L}(\mathscr{X}^n)/I_{\mathscr{P}_n}$ and $\mathscr{L}(\mathscr{N}^n)$.

Theorem 2 (Finite representation). *A choice function C on $\mathscr{L}(\mathscr{X}^n)$ is exchangeable if and only if there is a unique representing choice function $\tilde{C}$ on $\mathscr{L}(\mathscr{N}^n)$ such that*

$$C(A) = \{f \in A : \mathrm{H}_n(f) \in \tilde{C}(\mathrm{H}_n(A))\}$$

for all A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}^n))$. Furthermore, in that case, $\tilde{C}$ is given by $\tilde{C}(\mathrm{H}_n(A)) = \mathrm{H}_n(C(A))$ for all A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}^n))$. Finally, C is coherent if and only if $\tilde{C}$ is.

Is there a representation that does not depend on counts?

Polynomial gambles Consider the $\mathscr{X}$-simplex $\Sigma_{\mathscr{X}} := \{\theta \in \mathbb{R}^{\mathscr{X}}_{\geq 0} : \sum_{z\in\mathscr{X}} \theta_z = 1\}$, and the linear space $\mathscr{V}^n(\Sigma_{\mathscr{X}})$ of **polynomial gambles** h on $\Sigma_{\mathscr{X}}$ — the restrictions to $\Sigma_{\mathscr{X}}$ of multivariate polynomials p on $\mathbb{R}^{\mathscr{X}}$, in the sense that $h(\theta) = p(\theta)$ for all θ in $\Sigma_{\mathscr{X}}$.

Bernstein gambles For any n in $\mathbb{N}$ and any m in $\mathscr{N}^n$, let the **Bernstein basis polynomial** B_m on $\mathbb{R}^{\mathscr{X}}$ be given by $B_m(\theta) := \binom{n}{m} \prod_{x\in\mathscr{X}} \theta_x^{m_x}$ for all θ in $\mathbb{R}^{\mathscr{X}}$. The restriction to $\Sigma_{\mathscr{X}}$ is called a **Bernstein gamble**, which we also denote by B_m. $\{B_m : m \in \mathscr{N}^n\}$ is a basis for $\mathscr{V}^n(\Sigma_{\mathscr{X}})$.

Useful maps Consider the linear order isomorphisms

$$\mathrm{CoM}_n \colon \mathscr{L}(\mathscr{N}^n) \to \mathscr{V}^n(\Sigma_{\mathscr{X}}) : r \mapsto \sum_{m\in\mathscr{N}^n} r(m) B_m$$

and $\mathrm{M}_n := \mathrm{CoM}_n \circ \mathrm{H}_n$:

$$\mathrm{M}_n \colon \mathscr{L}(\mathscr{X}^n) \to \mathscr{V}^n(\Sigma_{\mathscr{X}}) : f \mapsto \mathrm{M}_n(f|\theta),$$

where $\mathrm{M}_n(f|\theta) := \sum_{m\in\mathscr{N}^n} \sum_{y\in[m]} f(y) \prod_{x\in\mathscr{X}} \theta_x^{m_x}$ is the expectation of f associated with the multinomial distribution whose parameters are n and θ.

Theorem 3 (Finite representation with polynomials). *A choice function C on $\mathscr{L}(\mathscr{X}^n)$ is exchangeable if and only if there is a unique representing choice function $\tilde{C}$ on $\mathscr{V}^n(\Sigma_{\mathscr{X}})$ such that*

$$C(A) = \{f \in A : \mathrm{M}_n(f) \in \tilde{C}(\mathrm{M}_n(A))\}$$

for all A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}^n))$. Furthermore, in that case, $\tilde{C}$ is given by $\tilde{C}(\mathrm{M}_n(A)) = \mathrm{M}_n(C(A))$ for all A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}^n))$. Finally, C is coherent if and only if $\tilde{C}$ is.

$$\begin{tikzcd}
\mathscr{L}(\mathscr{N}^n) \arrow[rr, leftrightarrow, "\mathrm{CoM}_n"] & & \mathscr{V}^n(\Sigma_{\mathscr{X}}) \arrow[dd, dotted] \\
& \mathscr{L}(\mathscr{X}^n) \arrow[ul, "\mathrm{H}_n"'] \arrow[ur, "\mathrm{M}_n"] \arrow[d, "{[\cdot]}"] & \\
& \mathscr{L}(\mathscr{X}^n)/I_{\mathscr{P}_n} \arrow[uul, leftrightarrow, "\tilde{\mathrm{H}}_n"] \arrow[uur, leftrightarrow, "\tilde{\mathrm{M}}_n"'] & \mathscr{V}(\Sigma_{\mathscr{X}})
\end{tikzcd}$$

What about the countable case?

Cylindrical extension In case of infinitely many exchangeable X_k, the global possibility space is $\mathscr{X}^{\mathbb{N}}$. We identify any gamble f on $\mathscr{X}^n$, with its **cylindrical extension**

$$f^*(x_1,\dots,x_n,\dots) := f(x_1,\dots,x_n)$$

for all $(x_1,\dots,x_n,\dots)$ in $\mathscr{X}^{\mathbb{N}}$. Using this convention, we can identify $\mathscr{L}(\mathscr{X}^n)$ with a subset of $\mathscr{L}(\mathscr{X}^{\mathbb{N}})$.

Gambles of finite structure We will call any gamble that depends only on a finite number of variables a **gamble of finite structure**. We collect all such gambles in

$$\overline{\mathscr{L}}(\mathscr{X}^{\mathbb{N}}) := \{f \in \mathscr{L}(\mathscr{X}^{\mathbb{N}}) : (\exists n \in \mathbb{N}) f \in \mathscr{L}(\mathscr{X}^n)\} = \bigcup_{n\in\mathbb{N}} \mathscr{L}(\mathscr{X}^n).$$

Set of indifferent gambles The subject assesses the sequence of variables $X_1, \dots, X_n, \dots$ to be exchangeable: he is indifferent between any gamble f in $\overline{\mathscr{L}}(\mathscr{X}^{\mathbb{N}})$ and its permuted variant $\pi^t f$, for any π in $\mathscr{P}_n$, where n now is the (finite) number of variables that f depends upon:

$$I_{\mathscr{P}} := \{f \in \overline{\mathscr{L}}(\mathscr{X}^{\mathbb{N}}) : (\exists n \in \mathbb{N}) f \in I_{\mathscr{P}_n}\} = \bigcup_{n\in\mathbb{N}} I_{\mathscr{P}_n}$$

is the subject's coherent set of indifferent gambles.

Countable exchangeability A choice function C on $\overline{\mathscr{L}}(\mathscr{X}^{\mathbb{N}})$ is called **(countably) exchangeable** if C is compatible with $I_{\mathscr{P}}$. C is exchangeable if and only if for every n in $\mathbb{N}$, its $\mathscr{X}^n$-marginal C_n is exchangeable. C_n is given by $C_n(A) := C(A)$ for all A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}^n))$.

We have an embedding: for every n in $\mathbb{N}$: $\mathscr{V}^n(\Sigma_{\mathscr{X}})$ is a linear subspace of $\mathscr{V}(\Sigma_{\mathscr{X}})$.

Theorem 4 (Countable Representation). *A choice function C on $\overline{\mathscr{L}}(\mathscr{X}^{\mathbb{N}})$ is exchangeable if and only if there is a unique representing choice function $\tilde{C}$ on $\mathscr{V}(\Sigma_{\mathscr{X}})$ such that, for every n in $\mathbb{N}$, the $\mathscr{X}^n$-marginal C_n of C is determined by*

$$C_n(A) = \{f \in A : \mathrm{M}_n(f) \in \tilde{C}(\mathrm{M}_n(A))\}$$

for all A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}^n))$. This $\tilde{C}$ is then given by $\tilde{C}(A) := \bigcup_{n\in\mathbb{N}} \tilde{C}_n(A \cap \mathscr{V}^n(\Sigma_{\mathscr{X}}))$ for all A in $\mathscr{Q}(\mathscr{V}(\Sigma_{\mathscr{X}}))$, with $\tilde{C}_n(\mathrm{M}_n(A)) := \mathrm{M}_n(C_n(A))$ for every A in $\mathscr{Q}(\mathscr{L}(\mathscr{X}^n))$, and where we let $\tilde{C}_n(\emptyset) := \emptyset$ for notational convenience. Finally, C is coherent if and only if $\tilde{C}$ is.

Can we add assessments?

Category permutation invariance Suppose that, in addition to exchangeability, the subject also has reason not to distinguish between the different elements of $\mathscr{X} := \{1,\dots,k\}$: consider any permutation ϖ of $\mathscr{X}$ and any outcome $X = (X_1,\dots,X_n)$ in $\mathscr{X}^n$, then he has reason not to distinguish between X and $\varpi(X) := (\varpi(X_1),\dots,\varpi(X_n))$. With any gamble f on $\mathscr{X}^n$ there corresponds a permuted gamble $\varpi^t f$, given by $(\varpi^t f)(x) = f(\varpi(x))$ for all x in $\mathscr{X}^n$.

Set of indifferent gambles Next to $I_{\mathscr{P}_n}$, also

$$I^{\mathrm{cat}} := \operatorname{span}\{f - \varpi^t f : f \in \mathscr{L}(\mathscr{X}^n), \varpi \in \mathscr{P}_{\mathscr{X}}\}$$

is a part of his set of indifferent gambles. Therefore, the smallest set of indifferent gambles compatible with this, is

$$I := \operatorname{span}\{I_{\mathscr{P}_n} \cup I^{\mathrm{cat}}\} = I_{\mathscr{P}_n} + I^{\mathrm{cat}}.$$

Proposition. *A coherent choice function C is compatible with I if and only if C is compatible with both $I_{\mathscr{P}_n}$ and I^{cat}.*

This defines a notion of **partition exchangeability** for choice functions. Instead of a representation in terms of count vectors, we now obtain a representation in terms of count vectors of count vectors.