SOS for bounded rationality

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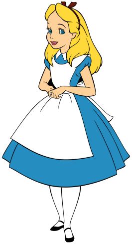
Theory of desirable gambles

We consider:

- **space of possibilities** Ω (finite or infinite)
- **gamble** $g : \Omega \to \mathbb{R}$ (bounded)
- some notation $\mathcal{L}$, $\mathcal{L}^+$, $\mathcal{L}^-$

$$\begin{array}{rcl} \Omega & = & \{HH, HT, TH, TT\} \\ \Omega & = & \mathbb{R}^2 \end{array} \qquad g(x_1, x_2)$$



Let $\mathcal{K} \subset \mathcal{L}$ the subset of the gambles that Alice finds desirable.

Theory of desirable gambles

How can we characterise the rationality of the assessments in $\mathcal{K}$?

Definition

We say that $\mathcal{K}$ is a coherent set of (almost) desirable gambles (ADG) when it satisfies the following rationality criteria:

- **A.1** If $\inf g > 0$ then $g \in \mathcal{K}$ (Accepting Sure Gains);
- **A.2** If $g \in \mathcal{K}$ then $\sup g \geq 0$ (Avoiding Sure Loss);
- **A.3** If $g \in \mathcal{K}$ then $\lambda g \in \mathcal{K}$ for every $\lambda > 0$ (Positive Scaling);
- **A.4** If $g, h \in \mathcal{K}$ then $g + h \in \mathcal{K}$ (Additivity);
- **A.5** If $g + \delta \in \mathcal{K}$ for every $\delta > 0$ then $g \in \mathcal{K}$ (Closure).

Note that A.1 and A.5 imply that $\mathcal{L}^+ \subseteq \mathcal{K}$ (including the zero gamble).

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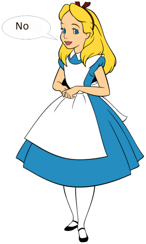
Note that A.1 and A.5 imply that $\mathcal{L}^+ \subseteq \mathcal{K}$ (including the zero gamble).

Duality: $$\mathcal{P} = \left\{ \mu \in \mathcal{M} : \mu \in \mathcal{M}^+,\ \int d\mu = 1,\ \int g d\mu \geq 0,\ \forall g \in \mathcal{K} \right\}$$

Content of the talk

$\Omega = \mathbb{R}^2$ $\qquad$ $g(x_1, x_2) = 4x_1^4 + 4x_1^3 x_2 - 3x_1^2 x_2^2 + 5x_2^4$?



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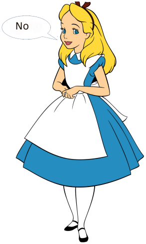

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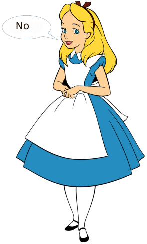

- Evaluating the **nonnegativity** of a gamble in infinite spaces is a difficult task.
- Instead of requiring the subject to accept all nonnegative gambles, we only require her to accept gambles for which she can efficiently determine the **nonnegativity**.
- We call this new criterion **bounded rationality**.

Towards a practical notion of desirability

Assume that the set of gambles that Alice finds to be desirable is finitely generated:

$$G = \{g_1, \dots, g_{|G|}\}$$

and so

$$\mathcal{K} = \text{posi}(G \cup \mathcal{L}^+)$$

where the posi of a set $A \subset \mathcal{L}$ is defined as

$$\text{posi}(A) := \left\{ \sum_{j=1}^{|G|} \lambda_j g_j \colon g_j \in A, \lambda_j \geq 0 \right\}$$

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By using this definition, it is clear that whenever $\mathcal{K}$ is finitely generated, it includes all nonnegative gambles and satisfies A.3, A.4 and A.5.

Towards a practical notion of desirability

Once Alice has defined G and so $\mathcal{K}$ via posi, ADG assumes that:

- she is able to check that $\mathcal{K}$ avoids sure loss (A.2 is also satisfied);
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It is easy to show that all above operations in ADG imply the assessment of the **nonnegativity** of a gamble.

Proposition (Natural Extension)

Given a finite set $G \subset \mathcal{L}$ of desirable gambles, the set $\mathrm{posi}(G \cup \mathcal{L}^+)$ *includes the gamble f if and only if there exist $\lambda_j \geq 0$ for $j = 1, \dots, |G|$ such that*

$$f - \sum_{j=1}^{|G|} \lambda_j g_j \geq 0.$$

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Lower Expectation (Prevision) of $h \Rightarrow f = h - \lambda_0$

Upper Expectation (Prevision) of $h \Rightarrow f = \lambda_0 - h$

Complexity

In case $\Omega = \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$

$$F := f - \sum_{j=1}^{|G|} \lambda_j g_j \overset{?}{\geq} 0$$

- In order to study the problem from a computational viewpoint, and avoid undecidability results, it is clear that we must impose further restrictions on the class of functions F
- At the same time we would like to keep the problem general enough, in order not to lose expressiveness of the model.

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- At the same time we would like to keep the problem general enough, in order not to lose expressiveness of the model.

A good compromise can be achieved by considering the case of **multivariate polynomials**.

The decidability of $F \geq 0$ for multivariate polynomials can be proven by means of the Tarski–Seidenberg quantifier elimination theorem.

Multivariate Polynomial gambles

Let $d \in \mathbb{N}$. By $\mathbb{R}_{2d}[x_1, \dots, x_n]$ we denote the set of all polynomials up to degree $2d$ in the indeterminate variables $x_1, \dots, x_n \in \mathbb{R}$ with real-valued coefficients.

Any polynomial in $\mathbb{R}_{2d}[x_1, \dots, x_n]$ can be written as

$$p(x_1, \dots, x_n) = b^\top \; v_{2d}(x_1, \dots, x_n)$$

with

$$v_{2d}(x_1, \dots, x_n) = [1, x_1, \dots, x_n, x_1^2, x_1 x_2, \dots, x_{n-1} x_n, x_n^2, \dots, x_1^{2d}, \dots, x_n^{2d}]^\top$$

$b \in \mathbb{R}^{s_n(2d)}$ with $s_n(j) = \binom{n+j}{j}$ for $j = 0, 1, 2, \dots$.

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Example

$$g(x_1, x_2) = 1 + x_1 + x_1^2 - 2x_2^2 = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix}$$

Gambles space

We fix d and we only consider gambles in

$$\mathcal{L}_{2d} = \mathbb{R}_{2d}[x_1, \dots, x_n], \quad \mathcal{L}_{2d}^{+} = \mathbb{R}_{2d}^{+}[x_1, \dots, x_n],$$

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SOS

SOS polynomials

The subset of polynomials

$$\Sigma_{2d}[x_1, \ldots, x_n] = \Big\{ p(x_1, \ldots, x_n) \in \mathbb{R}_{2d}[x_1, \ldots, x_n] \;\Big|\; p(x_1, \ldots, x_n) = v_d^\top(x_1, \ldots, x_n) Q v_d(x_1, \ldots, x_n) \text{ with } Q \in \mathbb{R}_s^{s_n(d) \times s_n(d)},\ Q \geq 0 \Big\}$$

where $\mathbb{R}_s^{s_n(d) \times s_n(d)}$ is the space of $s_n(d) \times s_n(d)$ real-symmetric matrices.

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SOS: Sum-Of-Squares Polynomials.

SOS polynomials are a subset of nonnegative polynomials:

$$\Sigma_{2d}[x_1,\ldots,x_n] \subset \mathbb{R}_{2d}^{+}[x_1,\ldots,x_n]$$

For instance $g(x_1,x_2) = x_1^2 x_2^2 (x_1^2 + x_2^2 - 1) + 1$ is a nonnegative polynomial that is not SOS.

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where $\Sigma_{2d} \subset \mathcal{L}^{+}_{2d}$ is the set of SOS of degree less than or equal to $2d$.

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Here, we restrict A.1 imposing bounded-rationality that implies that the set must only include SOS polynomials up to degree $2d$.

In BADG theory, we ask Alice only to accept SOS polynomials, i.e., gambles for which she can efficiently determine the nonnegativity.

Bounded rationality

Theorem (Bounded rationality Natural Extension)

Given a finite set $G \subset \mathcal{L}_{2d}$ *of desirable gambles, the set* $\operatorname{posi}(G \cup \Sigma_{2d})$ *includes the gamble* f *if and only if there exist* $\lambda_j \geq 0$ *for* $j = 1, \dots, |G|$ *such that*

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A BADG $\mathcal{C}$ that satisfies A.2 but not A.1 can (in theory) be turned to an ADG

- in $\mathcal{L}_{2d}$ by considering its extension $\mathrm{posi}(\mathcal{C} \cup \mathcal{L}^+_{2d})$
- in $\mathcal{L}$ by considering its extension $\mathrm{posi}(\mathcal{C} \cup \mathcal{L}^+)$

This is important because it will allow us to use BADG as a computable approximation of ADG.

Duality for BADG

Polynomials on $\mathbb{R}^n$ are not bounded functions.

The rationality criteria A.1–A.5 do not explicitly need boundedness, but boundedness is essential to show the duality between ADG and closed convex set of probability charges.

Since we are dealing with a **vector space**, we can consider its dual space $\mathcal{L}_{2d}^{\bullet}$, defined as the set of all linear maps $L : \mathcal{L}_{2d} \to \mathbb{R}$ (linear functionals).

The dual of $\mathcal{C} \subset \mathcal{L}_{2d}$ is defined as

$$\mathcal{C}^{\bullet} = \{L \in \mathcal{L}_{2d}^{\bullet} : L(g) \geq 0, \ \forall g \in \mathcal{C}\}.$$

Duality for BADG

Since $\mathcal{L}_{2d}$ has a basis, i.e., the monomials, if we introduce the scalars

$$y_{\alpha_1\alpha_2\ldots\alpha_n} := L(x_1^{\alpha_1}x_2^{\alpha_2},\ldots,x_n^{\alpha_n}) \in \mathbb{R},$$

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$L(g)$ for any polynomial g as a function of the real variables $y_{\alpha_1 \alpha_2 \ldots \alpha_n}$.

This means that $\mathcal{L}_{2d}^{\bullet}$ is isomorphic to $\mathbb{R}^{s_n(2d)}$.

This explains the **importance of bounding the degree** d of the polynomials.

Duality for BADG

We have then the following result[1]

Theorem

Let $\mathcal{C}$ be a BADG. Then its dual is

$$\mathcal{C}^{\bullet} = \left\{ y \in \mathbb{R}^{s_n(2d)} : M_{n,d}(y) \geq 0, \quad L(g) \geq 0, \quad \forall g \in \mathcal{C} \right\}$$

where $L(g)$ is determined by y, $M_{n,d}(y) := L(v_d(x_1, \dots, x_n) v_d(x_1, \dots, x_n)^\top)$.

Example

For instance, in the case $n = 1$ and $d = 2$, we have that

$$M_{1,2}(y) = L(v_2(x_1)v_2(x_1)^\top) = L\left(\begin{bmatrix} 1 & x_1 & x_1^2 \\ x_1 & x_1^2 & x_1^3 \\ x_1^2 & x_1^3 & x_1^4 \end{bmatrix} \right) = \begin{bmatrix} 1 & y_1 & y_2 \\ y_1 & y_2 & y_3 \\ y_2 & y_3 & y_4 \end{bmatrix}$$

[1] Jean Bernard Lasserre. *Moments, positive polynomials and their applications.* Vol. 1. World Scientific, 2009.

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$$\mathcal{P} = \left\{ \mu \in \mathcal{M} : \mu \in \mathcal{M}^+, \quad \int d\mu = 1, \int g d\mu \geq 0, \ \forall g \in \mathcal{K} \right\}$$

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$$y_{\alpha_1\alpha_2\ldots\alpha_n} = \int x_1^{\alpha_1} x_2^{\alpha_2}, \ldots, x_n^{\alpha_n} d\mu$$

as the expectation of $x_1^{\alpha_1} x_2^{\alpha_2}, \ldots, x_n^{\alpha_n}$ w.r.t. a **charge** μ.

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$$y_{\alpha_1\alpha_2\ldots\alpha_n} = \int x_1^{\alpha_1} x_2^{\alpha_2}, \ldots, x_n^{\alpha_n} d\mu$$

as the expectation of $x_1^{\alpha_1} x_2^{\alpha_2}, \ldots, x_n^{\alpha_n}$ w.r.t. a **charge** μ.
Note that, $y_0 = L(1) = 1$ implies that $\int 1 d\mu = 1$ under this interpretation.

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$$M_{1,2}(y) = L(v_2(x_1)v_2(x_1)^\top) = L\left(\begin{bmatrix} 1 & x_1 & x_1^2 \\ x_1 & x_1^2 & x_1^3 \\ x_1^2 & x_1^3 & x_1^4 \end{bmatrix}\right) = \begin{bmatrix} y_0 & y_1 & y_2 \\ y_1 & y_2 & y_3 \\ y_2 & y_3 & y_4 \end{bmatrix}$$

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The constraint $M_{n,d}(y) \geq 0$ is not strong enough to guarantee non-negativity of μ (it is only a necessary condition).

Negative probabilities are a manifestation of incoherence, that is they are a manifestation of the assumption of **bounded rationality**.

BADG as an approximating theory for ADG

We can use BADG as a computable approximating theory for ADG.

So let us consider the BADG set $\mathcal{C} = \text{posi}(G \cup \Sigma_{2d})$ and the corresponding ADG set $\mathcal{K} = \text{posi}(G \cup \mathcal{L}^{+})$ (same G).

Theorem

Assume that $\mathcal{K}$ avoids sure loss and let $f \in \mathcal{L}_{2d}$, then BADG is a conservative approximation of ADG theory in the sense that $\underline{P}^{*}(f) \leq \underline{P}(f)$.

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- it can be shown that $\underline{P}^*(f) \xrightarrow{d\to\infty} \underline{P}(f)$.[2]
- when G is empty (Alice is in a state of full ignorance)

$$\underline{P}[f] = \inf f$$

this explains why SOS polynomials are used in optimization, i.e., $\underline{P}^*[f]$ provides a lower bound for the minimum of f.

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Conclusions

- We have defined a theory of bounded rationality BADG.
- BADG provides an outer-approximation of ADG.
- We can also define an updating rule (conditioning) for BADG
  - note that in $\Sigma_{2d}[x_1, \ldots, x_n]$ there are not **Indicator Functions**.

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