Evenly Convex Credal Sets

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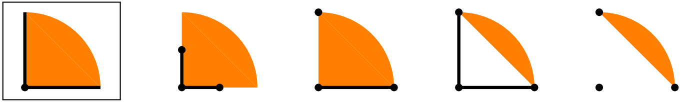
Goal

- There are many axiomatizations for closed credal sets, and very few for more general credal sets (... Teddy's work).

- Here we focus on *evenly* convex credal sets (intersection of open halfspaces).
  - That is, we want to allow $\mathbb{P}(A) > 1/2$ and $\mathbb{P}(A) \leq 2/3$.
  - Not the most general convex sets, but still quite flexible.



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- Focus on gambles over finite space Ω.
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  Cancellation: For all $\alpha \in (0, 1]$, $X \succ Y$ iff
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If the cone is an open set, then there is credal set $\mathcal{K}$ such that:

$X \in \mathcal{D}$ iff $\mathbb{E}_{\mathbb{P}}[X] > 0$ for all $\mathbb{P} \in \mathcal{K}$.

Other conditions

Aumann's continuity: If $\alpha X + (1-\alpha)Y \succ Z$ for all $\alpha > 0$, then either $Y \succ Z$ or $Y \not\sim Z$.

SSK-continuity If $X_i \succ Y_i$ for every i, and $\lim_i Y_i \succ Z$, then $\lim_i X_i \succ Z$, whenever limits exist.

Proposal: even continuity

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Even continuity If $Y \succ 0$ is false, then $\lim_i(\lambda_i Y - X_i) \succ 0$ is false for any sequence of $\lambda_i > 0$, $X_i \succ 0$ such that the limit exists.

That is: one cannot take an undesirable gamble Y and make it desirable, not even in the limit, by multiplying it by a positive number and subtracting from it a desirable gamble.

Main results

Definition

A preference ordering $\succ$ is *coherent* when it satisfies monotonicity, cancellation, and even continuity.

Theorem

If a preference ordering $\succ$ is coherent, then there is an evenly convex cone $\mathcal{D}$ of gambles, not containing the origin but containing the interior of the positive octant, such that $X \succ Y$ iff $X - Y \in \mathcal{D}$.

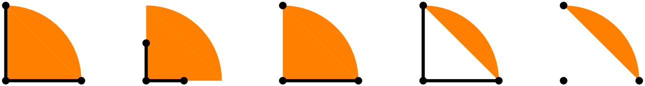
Theorem

If a preference ordering $\succ$ is coherent, then there is a unique maximal evenly convex credal set $\mathcal{K}$ such that $X \succ Y$ iff for all $\mathbb{P} \in \mathcal{K}$ we have $\mathbb{E}_{\mathbb{P}}[X] > \mathbb{E}_{\mathbb{P}}[Y]$.

A few other results

Theorem

Suppose $\succ$ is a coherent preference ordering, and the credal set $\mathcal{K}$ represents $\succ$. A credal set $\mathcal{K}'$ represents $\succ$ iff $\mathrm{eco}\mathcal{K}' = \mathcal{K}$.



Theorem

If $\mathcal{F}$ is a face of $\mathrm{cl}\mathcal{D}$, *and $\mathcal{F} \cap \mathcal{D} \neq \emptyset$, then $\mathcal{F}^{\triangle} \cap \mathcal{C} = \emptyset$.*

Is this necessary?

- If $\succ$ is a coherent preference ordering, then SSK-continuity holds.
- Does SSK-continuity actually imply even equivalence?

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- If $\succ$ is a coherent preference ordering, then SSK-continuity holds.
- Does SSK-continuity actually imply even equivalence?
- YES in an important case! But NO in general.

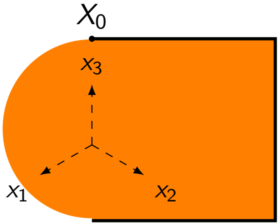
Theorem

Suppose $\succ$ is a preference ordering satisfying monotonicity, cancellation, and SSK-continuity, with representing set of desirable gambles $\mathcal{D}$. If the closure of $\mathcal{D}$ is the intersection of finitely many closed halfspaces, then $\mathcal{D}$ is evenly convex.

But not in general...

Example

$\Omega = \{\omega_1, \omega_2, \omega_3\}$; $\mathcal{B}$ as the union of the *open* circle with center $(1/4, 1/4, 1/2)$ and radius $\sqrt{3/2}$ drawn on the simplex consisting of $x_1 + x_2 + x_3 = 1$, and the *closed* polygon with four vertices $(3/4, 3/4, -1/2)$, $(-1/4, -1/4, 3/2)$, $(-2, 3/2, 3/2)$, $(-1, 5/2, -1/2)$; $X_0 = (-1/4, -1/4, 3/2)$, a non-exposed extreme point of $\mathcal{B}$. The cone $\mathcal{D}''$ is the set of all rays emanating from the origin and going through points of $\mathcal{B}$ except X_0; it produces a preference ordering that satisfies SSK-continuity.



Conclusion

- Presented a few axioms on preference orderings that, together, imply a representation through evenly convex credal sets.
  - Novel Archimedean condition (even continuity) that implies even convexity.
- A similar representation can be obtained using SSK-continuity in many, but not all, cases.

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