

Independent Natural Extension for Infinite Spaces

Williams-coherence
to the Rescue!



Jasper De Bock

Ghent University
Belgium











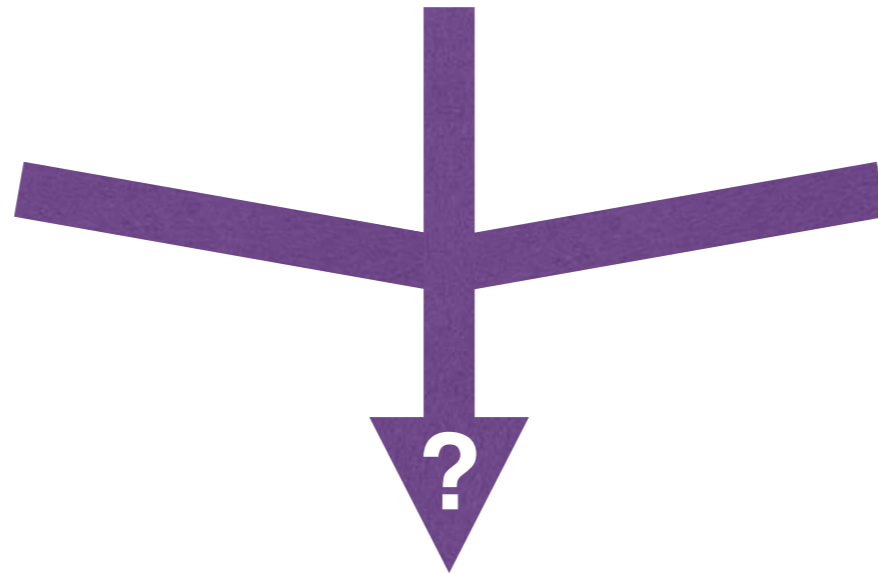
X_1

local
uncertainty
model

independent

X_2

local
uncertainty
model



joint uncertainty model

$$P(X_1|X_2) = P(X_1)$$

$$P(X_2|X_1) = P(X_2)$$

X_1

X_2

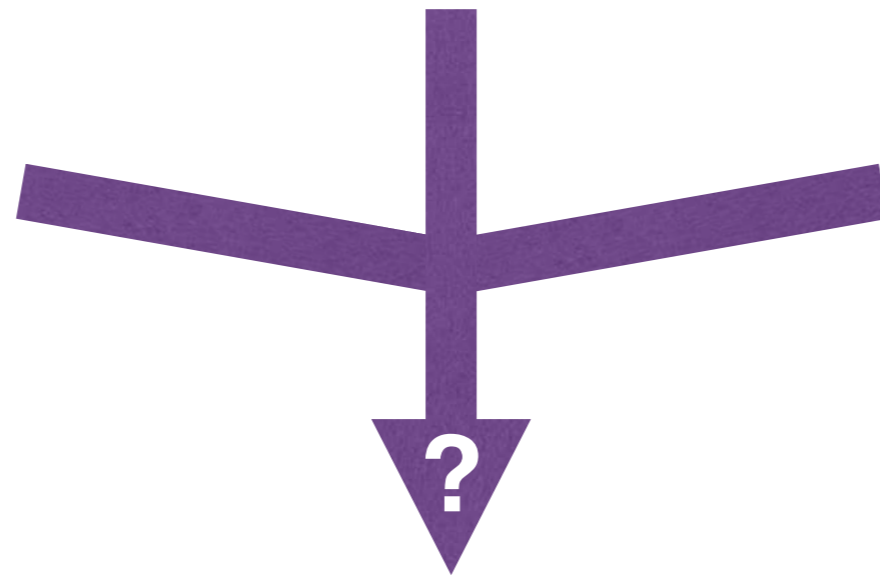
independent

**local
uncertainty
model**

**local
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$$P(X_1)$$

$$P(X_2)$$



joint uncertainty model

$$P(X_1, X_2)$$

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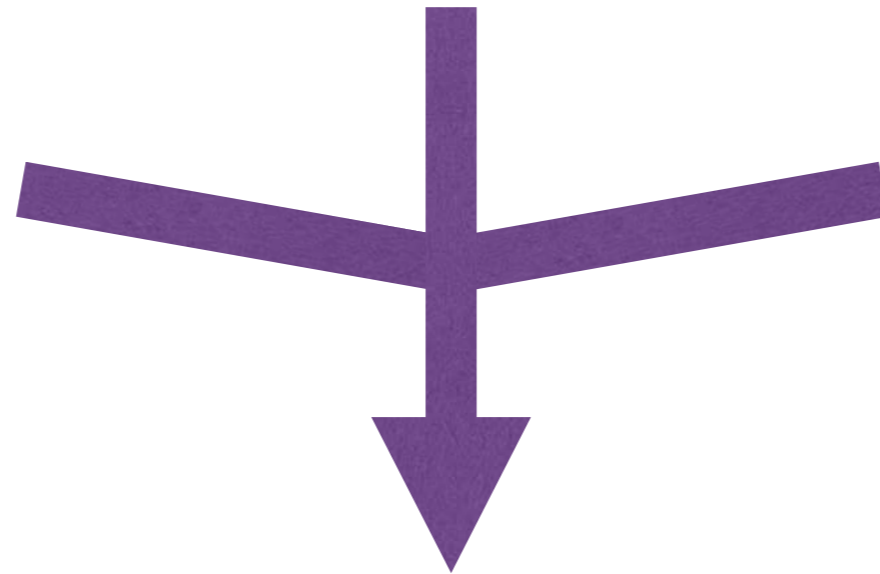
X_2

**local
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model**

$$P(X_1)$$

**local
uncertainty
model**

$$P(X_2)$$



joint uncertainty model

$$P(X_1, X_2) = P(X_1)P(X_2)$$

X_1

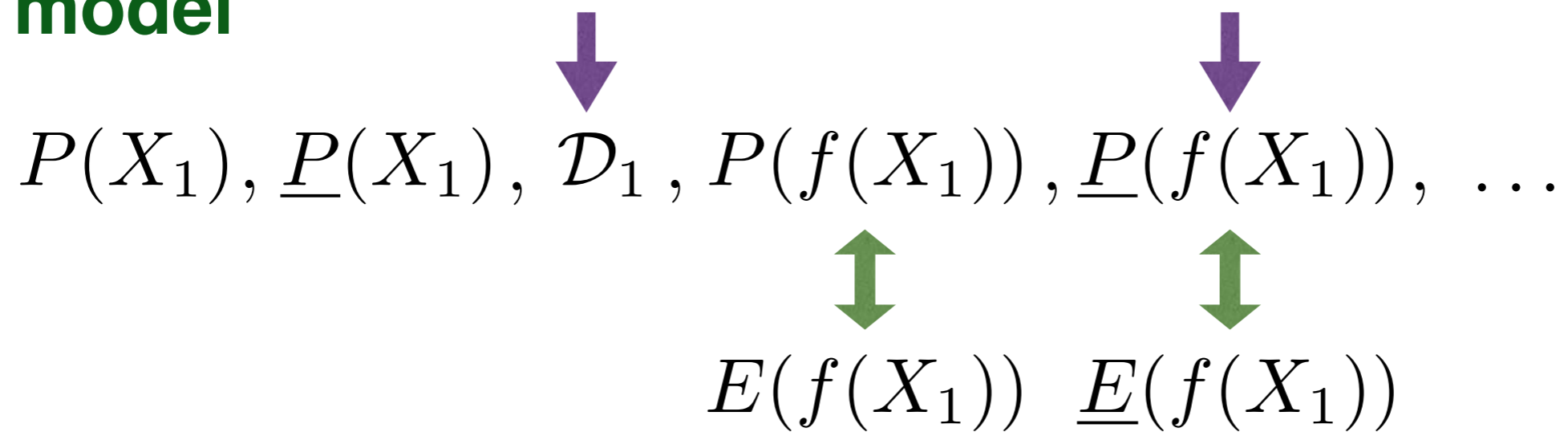
**local
uncertainty
model**

$P(X_1), \underline{P}(X_1), \mathcal{D}_1, P(f(X_1)), \underline{P}(f(X_1)), \dots$

\updownarrow \updownarrow
 $E(f(X_1))$ $\underline{E}(f(X_1))$

X_1

**local
uncertainty
model**



X_1

local
uncertainty
model

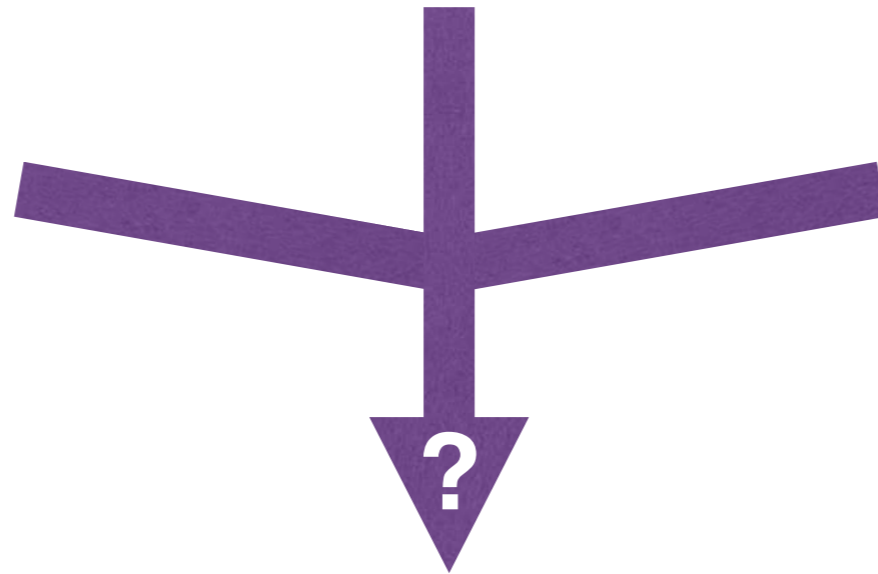
$$\underline{P}(f(X_1))$$

independent

X_2

local
uncertainty
model

$$\underline{P}(f(X_2))$$



joint uncertainty model

$$\underline{P}(f(X_1, X_2))$$

?

X_1

independent

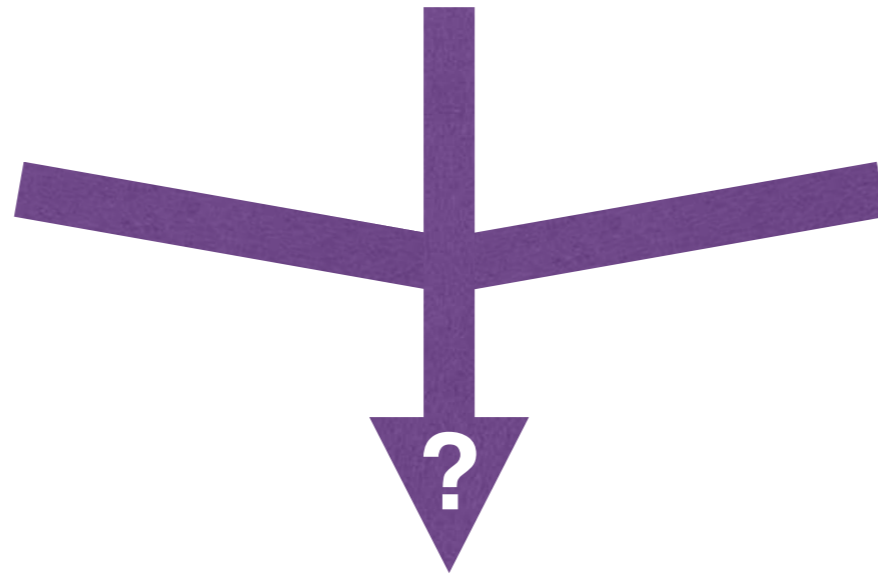
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joint uncertainty model

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X_1

X_2

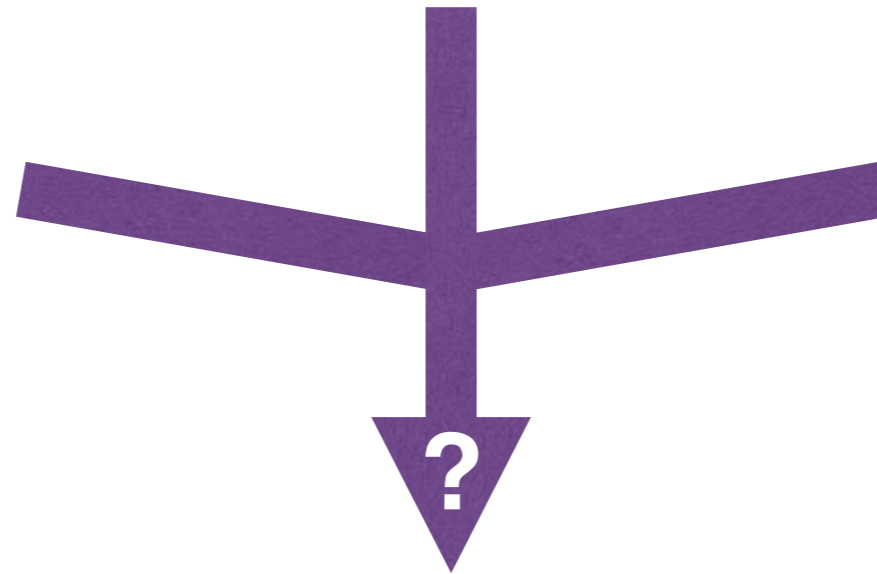
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joint uncertainty model

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X_1

independent

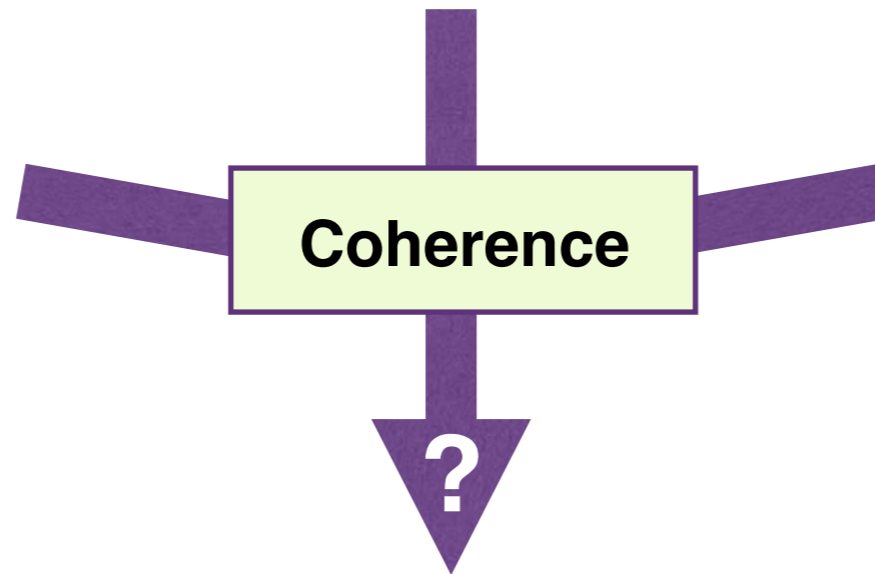
X_2

**local
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model**

**local
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model**

$$\underline{P}(f(X_1))$$

$$\underline{P}(f(X_2))$$



joint uncertainty model

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X_2

independent

**local
uncertainty
model**

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$$\underline{P}(f(X_1))$$

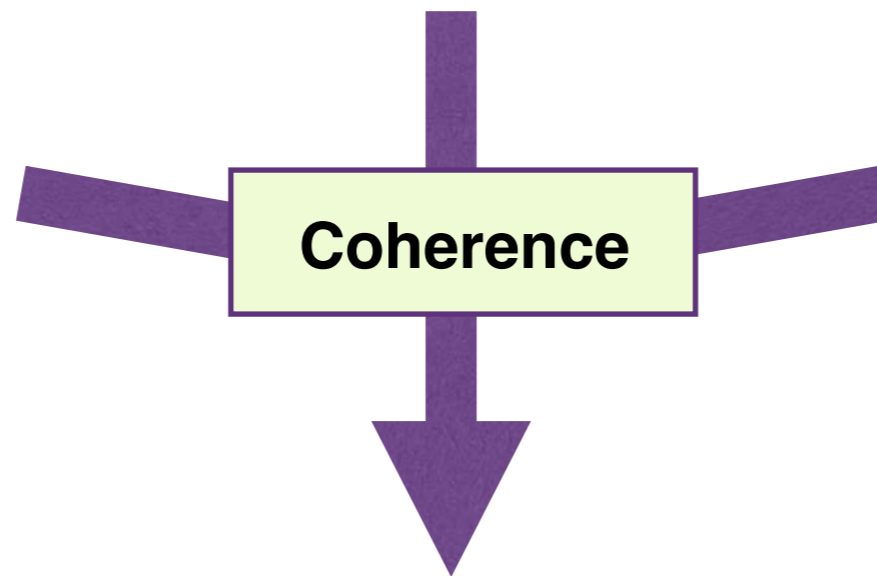
$$\parallel$$

$$\underline{P}_1(f)$$

$$\underline{P}(f(X_2))$$

$$\parallel$$

$$\underline{P}_2(f)$$



joint uncertainty model

$$(\underline{P}_1 \otimes \underline{P}_2)(f(X_1, X_2))$$

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Two very useful properties

External additivity

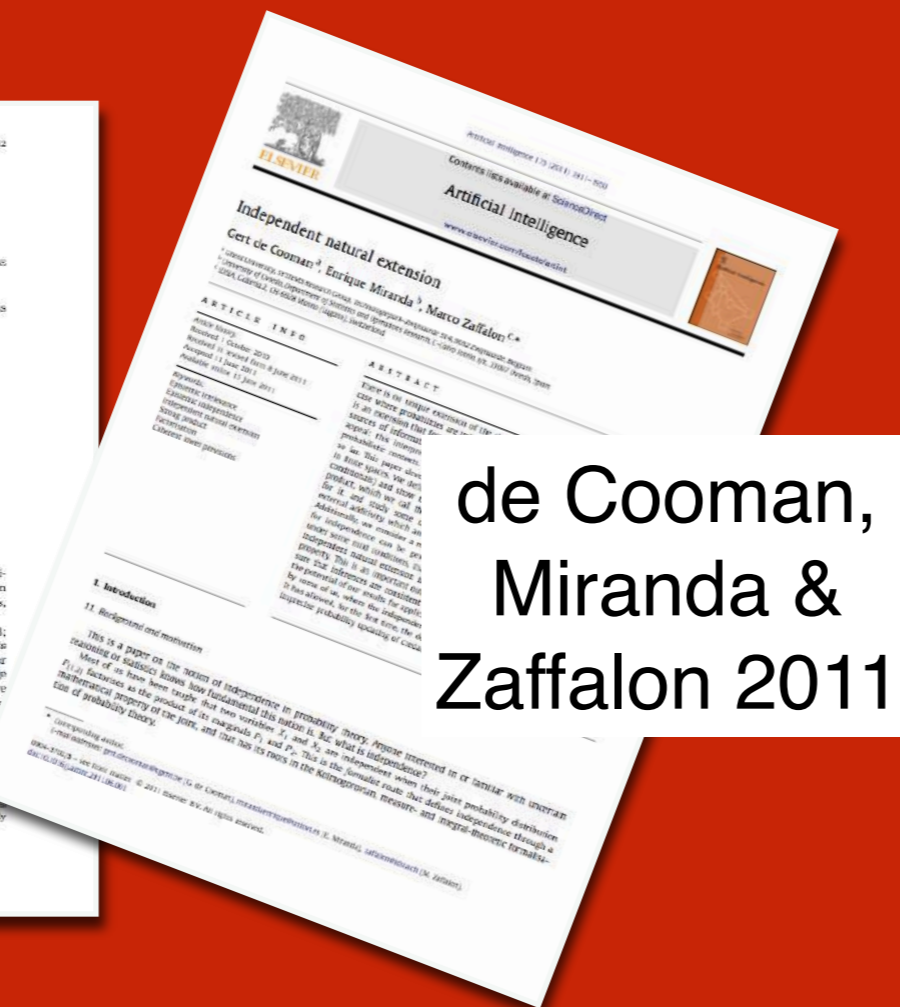
$$(\underline{P}_1 \otimes \underline{P}_2)(f(X_1) + h(X_2)) = \underline{P}_1(f(X_1)) + \underline{P}_2(h(X_2))$$

Factorisation

$$\begin{aligned} (\underline{P}_1 \otimes \underline{P}_2)(g(X_1)h(X_2)) &= \begin{cases} \underline{P}_1(g(X_1))\underline{P}_2(h(X_2)) & \text{if } \underline{P}(h(X_2)) \geq 0 \\ \bar{P}_1(g(X_1))\underline{P}_2(h(X_2)) & \text{if } \underline{P}(h(X_2)) \leq 0 \end{cases} \\ &\qquad \qquad \qquad \text{if } g \geq 0 \end{aligned}$$

DISCLAIMER!

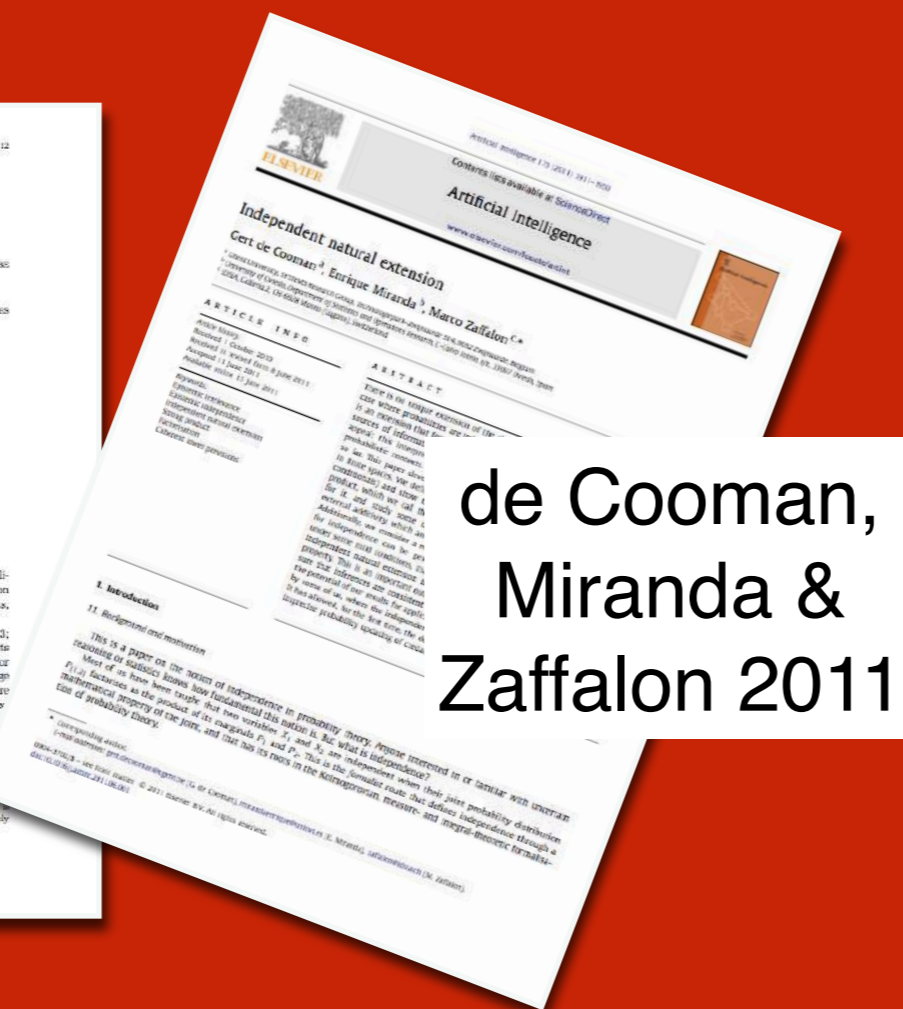
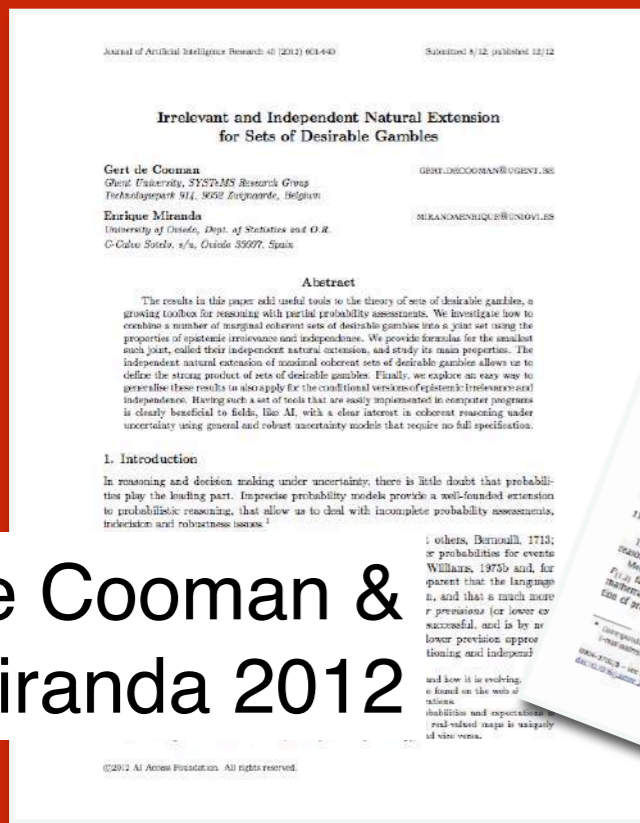
All of this is well known, and has been for several years now...



de Cooman,
Miranda &
Zaffalon 2011

DISCLAIMER!

All of this is well known, and has been for several years now...



de Cooman,
Miranda &
Zaffalon 2011

...but
only for
finite spaces!

Independent Natural Extension for **Infinite** Spaces



?



$$\underline{P}(f(X_1)|X_2) = \underline{P}(f(X_1))$$
$$\underline{P}(f(X_2)|X_1) = \underline{P}(f(X_2))$$

X_1

independent

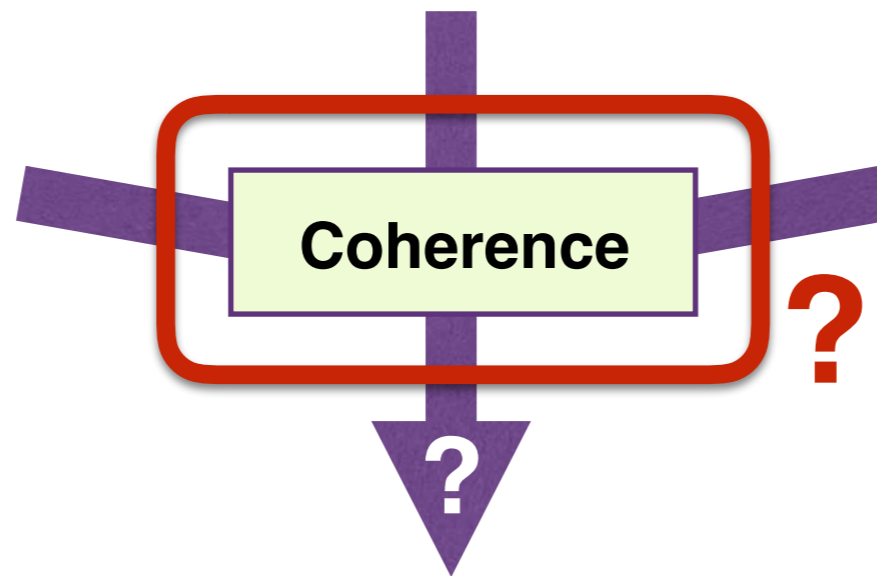
X_2

**local
uncertainty
model**

**local
uncertainty
model**

$$\underline{P}(f(X_1))$$

$$\underline{P}(f(X_2))$$



joint uncertainty model



Coherence

?

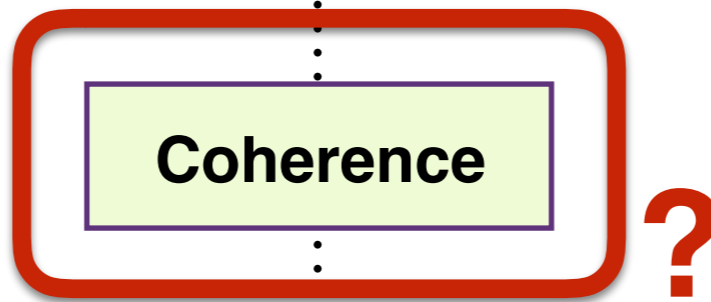
Walley ↔ Williams



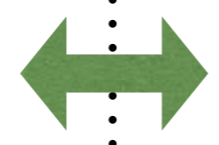
Independent natural extension may not exist!



Miranda & Zaffalon 2015



~~Walley~~



Williams

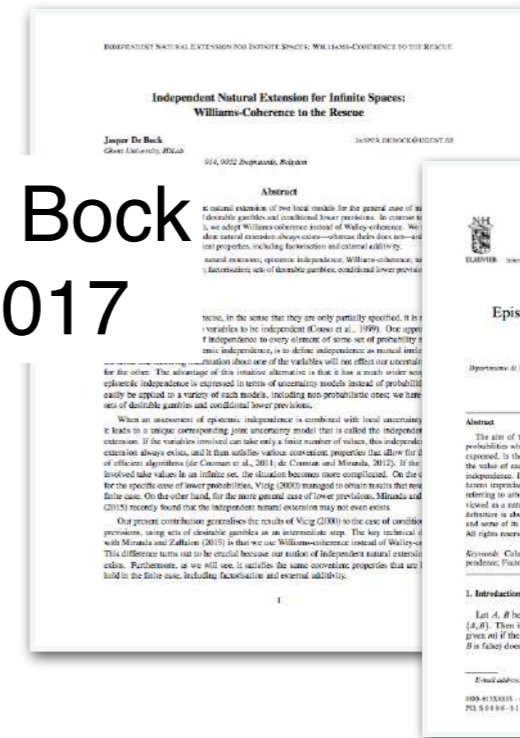


Independent natural extension may not exist!

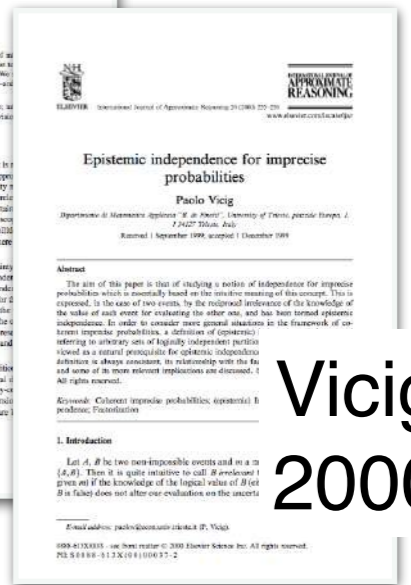


Miranda & Zaffalon 2015

Independent natural extension always exists!



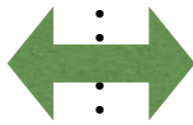
De Bock 2017



Vicig 2000

Coherence

~~Walley~~



Williams ✓



Independent Natural Extension for Infinite Spaces

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to the Rescue!**



Two very useful properties

External additivity ?

$$(\underline{P}_1 \otimes \underline{P}_2)(f(X_1) + h(X_2)) = \underline{P}_1(f(X_1)) + \underline{P}_2(h(X_2))$$

Factorisation ?

$$\begin{aligned} & (\underline{P}_1 \otimes \underline{P}_2)(g(X_1)h(X_2)) \\ &= \begin{cases} \underline{P}_1(g(X_1))\underline{P}_2(h(X_2)) & \text{if } \underline{P}(h(X_2)) \geq 0 \\ \bar{P}_1(g(X_1))\underline{P}_2(h(X_2)) & \text{if } \underline{P}(h(X_2)) \leq 0 \end{cases} \\ & \qquad \qquad \qquad \text{if } g \geq 0 \end{aligned}$$

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$$\underline{P}(f(X_1)|X_2) = \underline{P}(f(X_1))$$

$$\underline{P}(f(X_2)|X_1) = \underline{P}(f(X_2))$$

X_1

X_2

independent

**local
uncertainty
model**

**local
uncertainty
model**

$$\underline{P}(f(X_1))$$

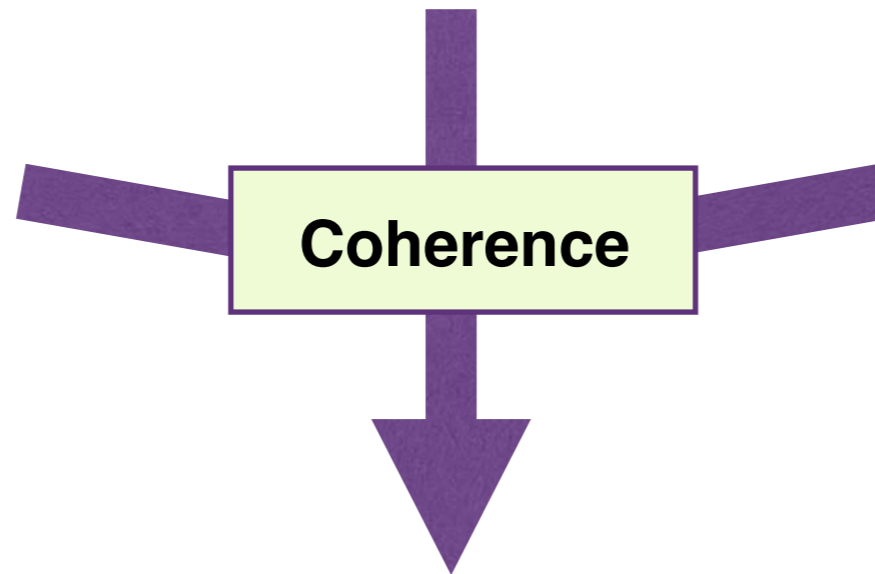
$$\parallel$$

$$\underline{P}_1(f)$$

$$\underline{P}(f(X_2))$$

$$\parallel$$

$$\underline{P}_2(f)$$

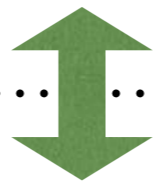


joint uncertainty model

$$(\underline{P}_1 \otimes \underline{P}_2)(f(X_1, X_2))$$

$$\begin{aligned} \underline{P}(f(X_1)|X_2) &= \underline{P}(f(X_1)) \\ \underline{P}(f(X_2)|X_1) &= \underline{P}(f(X_2)) \end{aligned}$$

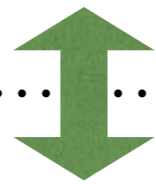
independent



$$\begin{aligned} \underline{P}(f(X_1)|B_2) &= \underline{P}(f(X_1)) \quad \forall B_2 \in \mathcal{B}_2 \\ \underline{P}(f(X_2)|B_1) &= \underline{P}(f(X_2)) \quad \forall B_1 \in \mathcal{B}_1 \end{aligned}$$

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independent



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value-independence: $\mathcal{B}_i = \{\{x_i\} : x_i \in \mathcal{X}_i\}$

subset-independence: $\mathcal{B}_i = \mathcal{P}(\mathcal{X}_i) \setminus \{\emptyset\}$

Two very useful properties

External additivity ✓

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Two very useful properties

External additivity ✓

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Factorisation ✓

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if $g \geq 0$ is \mathcal{B}_1 -measurable

**Independent
natural extension
may not exist!**

~~Walley~~



Williams



**Independent
natural extension
always exists!**

~~value-
independence~~



**subset-
independence**



**Factorisation
may not hold!**


**Factorisation
always holds!**

See you at the poster?



Independent Natural Extension for Infinite Spaces

Williams-Coherence to the Rescue!



Jasper De Bock
jasper.debock@ugent.be
Ghent University, Belgium

If you are not familiar with sets of desirable gambles, lower previsions, Williams-coherence, epistemic independence or independent natural extension, this poster may make little sense at first. I will do my very best to compensate with enthusiasm! If I fail, we can also simply go for a beer. In any case, the thought bubbles below may serve as a nice discussion starter.

Modelling Uncertainty

A subject's uncertainty about a variable X that takes values x in a—possibly infinite—set \mathcal{X} can be modelled in various ways. We consider two very general and closely connected frameworks, the latter of which includes probabilities as a special case.

Sets of desirable gambles. The basic idea here is to consider the subject's attitude towards gambles on \mathcal{X} , which are bounded real-valued functions f on \mathcal{X} whose value $f(x)$ represents the—possibly negative—payoff for the outcome x . In particular, we consider the gambles that she finds desirable, in the sense that she prefers them over not betting at all. We gather all these gambles in a so-called set of desirable gambles \mathcal{D} , which is a subset of the set $\mathcal{G}(\mathcal{X})$ of all gambles.

Conditional lower previsions. Here too, the idea is to model a subject's uncertainty about X by considering her attitude towards gambles on \mathcal{X} . However, in this case, instead of considering sets of gambles, we consider the prices at which a subject is willing to buy these gambles. Let $\mathcal{C}(\mathcal{X})$ be the set of all pairs (f, B) , where f is a gamble on \mathcal{X} and B is a non-empty subset of \mathcal{X} —an event. A conditional lower prevision P on a domain $\mathcal{C} \subseteq \mathcal{C}(\mathcal{X})$ is then a map $P: \mathcal{C} \rightarrow \bar{\mathbb{R}}: (f, B) \mapsto P(f|B)$, where $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. For any (f, B) in \mathcal{C} , the prevision $P(f|B)$ of f conditional on B is interpreted as the subject's supremum price μ for buying f , provided that the transaction is cancelled if B does not happen. In other words, $P(f|B)$ is the supremum value of μ for which she is eager to engage in a transaction where she receives $f(x) - \mu$ if $x \in B$ and zero otherwise. If $B = \mathcal{X}$, we write $P(f) := P(f|\mathcal{X})$ and call $P(f)$ the lower prevision of f .

Their connection. These two uncertainty frameworks are closely connected. In particular, because of their interpretation in terms of buying prices for gambles, a set of desirable gambles \mathcal{D} can be easily derived from a conditional lower prevision $P_{\mathcal{D}}$ on $\mathcal{C}(\mathcal{X})$, the corresponding conditional lower prevision $P_{\mathcal{D}}$ is defined by $P_{\mathcal{D}}(f|B) := \sup\{\mu \in \mathbb{R} : [f - \mu]_B \in \mathcal{D}\}$ for every $(f, B) \in \mathcal{C}(\mathcal{X})$.

Coherence. For an uncertainty model to represent a rational subject's beliefs, it needs to satisfy a set of rationality criteria; if it does, it is called coherent. For a set of desirable gambles \mathcal{D} , coherence means that for any gambles $f, g \in \mathcal{G}(\mathcal{X})$ and any real number $\lambda > 0$:

- D1. if $f \geq 0$ and $f \neq 0$, then $f \in \mathcal{D}$
- D2. if $f \in \mathcal{D}$ then $\lambda f \in \mathcal{D}$
- D3. if $f, g \in \mathcal{D}$, then $f + g \in \mathcal{D}$
- D4. if $f \leq 0$, then $f \notin \mathcal{D}$

A conditional lower prevision P on a domain $\mathcal{C} \subseteq \mathcal{C}(\mathcal{X})$ is then said to be coherent if there is a coherent set of desirable gambles \mathcal{D} on \mathcal{X} such that P coincides with $P_{\mathcal{D}}$ on \mathcal{C} . Equivalently, P is coherent if it satisfies the structure-free notion of Williams-coherence that was developed by Pelessoni and Vicig (2009).

All of this seems very abstract. Does it have any practical use?

That's weird! Shouldn't the right-hand side be unconditional?

You said that probabilities are a special case. Yeah right... how does that work?

So why is there no B_i here?

What happens if there are more than two variables?

Modelling Independence

We say that X_1 and X_2 are independent if our uncertainty model for X_1 is not affected by conditioning on information about X_2 , and vice versa. This information can easily be applied to a probability measure, and then yields the usual notion of independence. More generally, it can just as easily be applied to lower previsions, type of uncertainty model, and is then referred to as epistemic independence.

We consider a very general definition of epistemic independence. In particular, for every $i \in \{1, 2\}$, we consider any set of conditioning events \mathcal{C}_i for the variable X_i , that is, any subset of the set $\mathcal{C}_i(\mathcal{X}_i)$ of all non-empty subsets of \mathcal{X}_i .

A coherent conditional lower prevision P on $\mathcal{C}(\mathcal{X}_1 \times \mathcal{X}_2)$ is then called epistemically independent if for any i and j such that $\{i, j\} = \{1, 2\}$:

$$P(f_i|B_i, f_j|B_j) = P(f_i|B_i)$$

for all $(f_i, B_i) \in \mathcal{C}_i(\mathcal{X}_i)$ and $B_j \in \mathcal{C}_j$.

Similarly, a coherent set of desirable gambles \mathcal{D} on $\mathcal{X}_1 \times \mathcal{X}_2$ is epistemically independent if for any i and j such that $\{i, j\} = \{1, 2\}$ and for any $B_j \in \mathcal{C}_j$ that $\text{marg}_j(\mathcal{D}|B_j) = \text{marg}_j(\mathcal{D})$ in the sense that for all $f \in \mathcal{G}(\mathcal{X}_i)$:

$$f(X_i) \mathbb{1}_{B_j}(X_j) \in \mathcal{D} \Leftrightarrow f(X_i) \in \mathcal{D}$$

where $\mathbb{1}_{B_j}(X_j)$ is the indicator of B_j , defined by $\mathbb{1}_{B_j}(x_j) := 1$ if $x_j \in B_j$ and $\mathbb{1}_{B_j}(x_j) := 0$ otherwise.

Two special cases are particularly important. If $\mathcal{C}_1 = \mathcal{C}_1$ and $\mathcal{C}_2 = \mathcal{C}_2$, we obtain the special case of epistemic value-independence, which is the most commonly called epistemic independence. If obtain what we call epistemic subset-independence. As we will see, the latter has superior properties.

Independent Natural Extension

For all $i \in \{1, 2\}$, let \mathcal{D}_i be a local coherent set of desirable gambles on \mathcal{X}_i . The independent natural extension of \mathcal{D}_1 and \mathcal{D}_2 is then the smallest—most conservative—epistemically independent coherent set of desirable gambles on $\mathcal{X}_1 \times \mathcal{X}_2$ that extends them, meaning that

$$(\forall i \in \{1, 2\}) \mathcal{D}_i = \text{marg}_i(\mathcal{D}) = \{f \in \mathcal{G}(\mathcal{X}_i) : f(X_i) \in \mathcal{D}_i\}$$

For lower previsions, the local models P_1 and P_2 are coherent conditional lower previsions on $\mathcal{C}_1 \subseteq \mathcal{C}(\mathcal{X}_1)$ and $\mathcal{C}_2 \subseteq \mathcal{C}(\mathcal{X}_2)$, respectively. The independent natural extension of P_1 and P_2 is then the smallest—most conservative—epistemically independent coherent lower prevision on $\mathcal{C}(\mathcal{X}_1 \times \mathcal{X}_2)$ that extends them, meaning that

$$(\forall i \in \{1, 2\}) P_i(f_i|B_i) = P(f_i|B_i) \text{ for all } (f_i, B_i) \in \mathcal{C}_i$$

Existence. In both of our two frameworks, the independent natural extension always exists; we denote it by $\mathcal{D}_1 \otimes \mathcal{D}_2$ and $P_1 \otimes P_2$, respectively. For lower previsions, this result crucially depends on our use of Williams-coherence; for Walley-coherence, as shown by Mirandis and Zaffalon (2015) for epistemic value-independence, this may no longer hold.

Properties. Let $\{i, j\} = \{1, 2\}$ and consider any $h \in \mathcal{G}(\mathcal{X}_j)$ and $f, g \in \mathcal{G}(\mathcal{X}_i)$ such that $f \geq 0$ is \mathcal{D}_i -measurable—a technical condition that coincides with the usual notion when $\mathcal{C}_i \cup \{\emptyset\}$ is a σ -field. Then all the terms are well-defined—if \mathcal{C}_1 and \mathcal{C}_2 are large enough—we have that

$$(P_1 \otimes P_2)(f + gh) = P_1(f + gP_2(h))$$

As a direct consequence, we find that

$$(P_1 \otimes P_2)(f + h) = P_1(f) + P_2(h)$$

and—with $P_i(h) = -P_i(-h)$ —that

$$(P_1 \otimes P_2)(gh) = P_1(gP_2(h)) = \begin{cases} P_1(g)P_2(h) & \text{if } P_2(h) \geq 0 \\ P_1(g)P_2(h) & \text{if } P_2(h) \leq 0 \end{cases}$$

known as external additivity and factorisation, respectively. Crucially, for epistemic subset-independence, \mathcal{D}_i -measurability is trivially satisfied, and factorisation then always holds.