Game solutions, probability transformations and the core

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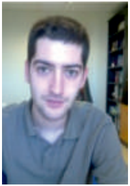

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Summary

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\begin{tikzcd}
\boxed{\text{Game theory}} \arrow[d] & \boxed{\nu : \mathcal{P}(\Omega) \to [0,1]} \arrow[l] \arrow[r] & \boxed{\text{Imprecise probabilities}} \arrow[d] \\
\boxed{\text{Distributions of the payoff}} \arrow[d] & \boxed{\mathcal{M}(\nu) = \{P \text{ additive} : P \geq \nu\}} \arrow[l] \arrow[r] & \boxed{\text{Compatible probabilities}} \arrow[d] \\
\boxed{\text{Optimal distribution}} & \boxed{\text{Shapley value}} \arrow[l] \arrow[r] & \boxed{\text{Representative}}
\end{tikzcd}
$$

Game theory with transferable utility

Consider a monotone and normalized game with a set of players Ω, represented by means of a function $\nu : \mathcal{P}(\Omega) \to [0,1]$.

A subset $A \subset \Omega$ represents a coalition of players, and $\nu(A)$ is the gain that is guaranteed to the players in A.

$\mathcal{M}(\nu) = \{P$ probability measure $: P(A) \geq \nu(A)\}$ is the set of distributions of the total payoff that cannot be improved by any coalition. It is called the core of the game.

We assume transferable utility.

Solutions of a game

A solution concept of a game is a function that gives the optimal distribution of the total payoff among the players, according to some desirable properties.

The most important one is Shapley value, that is given by:

$$\Phi(\nu)(i) = \sum_{T \not\supseteq \{i\}} \frac{t!(n-t-1)!}{n!} (\nu(T \cup \{i\}) - \nu(T)),$$

where $t = |T|$ and n is the number of players.

Banzhaf value

Another important solution is the Banzhaf value:

$$B(\nu)(i) = \frac{1}{2^{n-1}} \sum_{T \not\supseteq \{i\}} \nu(T \cup \{i\}) - \nu(T).$$

Unlike Shapley value, this solution is not normalized: we may have $\sum_i B(\nu)(i) \neq 1$. This led van der Brink and van der Laan to propose the normalized Banzhaf value:

$$\Psi(\nu)(i) = \frac{B(\nu)(i)}{\sum_j B(\nu)(j)}.$$

Imprecise probabilities

In imprecise probability theory, we can consider a random variable X with values in a set Ω. Then we regard $\nu : \mathcal{P}(\Omega) \to [0,1]$ as a model for the imprecise knowledge for the distribution of X.

Then $\nu(A)$ is a lower bound for the probability that X takes a value in A, and $\mathcal{M}(\nu) = \{P \text{ probability measure } : P(A) \geq \nu(A)\}$ would be the set of probabilities that are compatible with the available information.

Depending on the properties ν satisfies, we talk of different types of imprecise probability models.

Connection game theory $\leftrightarrow$ imprecise probabilities

Shapley showed that, when ν is convex (=2-monotone)

$$\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B) \ \forall A, B,$$

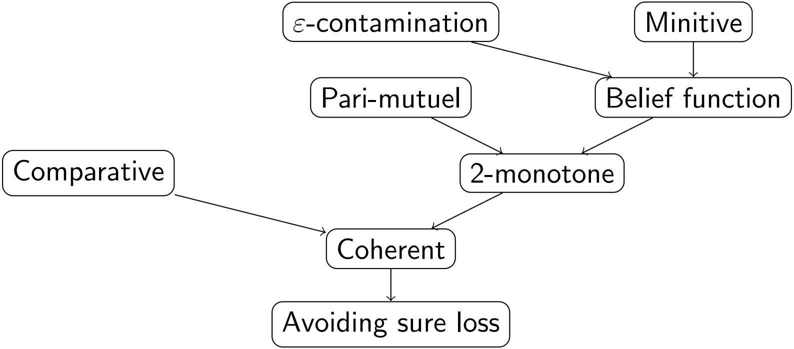
Shapley value coincides with the center of gravity of $\mathcal{M}(\nu)$.

Thus, $\Phi(\nu)$ is the average of the extreme points of $\mathcal{M}(\nu)$, and this implies that it coincides with the pignistic transformation of ν, proposed among others by Smets.

This connects game solutions with the problem of probability transformations: transforming a IP model ν into a representing probability measure in $\mathcal{M}(\nu)$.

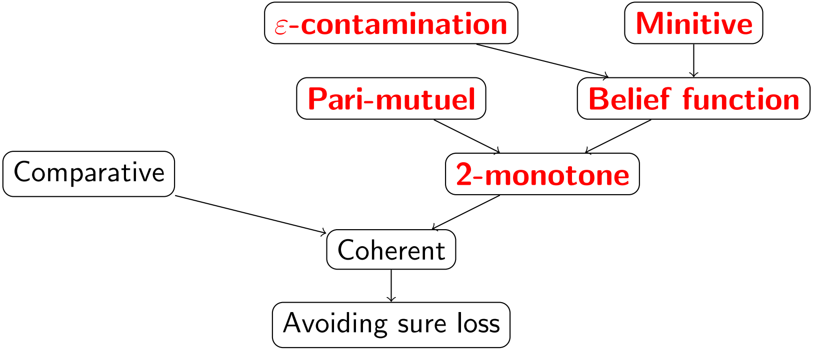
Relationships between the IP models

We investigate the role of Shapley and Banzhaf values as probability transformations for a number of different IP models:



Particular cases of 2-monotone games

First we consider a number of particular cases of 2-monotone capacities:



Belief functions

A 2-monotone capacity is a belief function when its Möbius inverse m is non-negative.

- $\Psi(\nu)$ need not belong to $\mathcal{M}(\nu)$.
- When ν is the unanimity game on S, $\nu(A) = \begin{cases} 1 \text{ if } S \subseteq A \\ 0 \text{ otherwise} \end{cases}$, we have that $\Phi(\nu) = \Psi(\nu)$ is the uniform distribution on S.

Two particular cases of belief functions are *minitive measures* and *ε-contamination models*.

Minitive measures

ν is minitive when $\nu(A \cap B) = \min\{\nu(A), \nu(B)\}\ \forall A, B \subseteq \mathcal{X}$.

- $\Phi(\nu)(i) = \sum_{j=i}^{n} \frac{m(\{1,\ldots,j\})}{j} \quad \forall i$ (Dubois and Prade).
- $\Psi(\nu)(i) = \frac{\sum_{j=i}^{n} 2^{n-j} m(\{1,\ldots,j\})}{\sum_{j=1}^{n} j \cdot 2^{n-j} m(\{1,\ldots,j\})} \quad \forall i$.
- $\sum_i B(\nu)(i) \leq 1$.

ε-contamination models

Given a probability measure P_0 and $\varepsilon \in (0,1)$, the ε-contamination model they induce is

$$\nu(A) = (1-\varepsilon)P_0(A) + \varepsilon\nu_\Omega(A),$$

where $\nu_\Omega(A) = 0$ unless $A = \Omega$ (the unanimity game on Ω).

- $\Phi(\nu)(i) = (1-\varepsilon)P_0(\{i\}) + \frac{\varepsilon}{n}\ \forall i$.
- $\Psi(\nu)(i) = \frac{(1-\varepsilon)P_0(\{i\}) + \frac{\varepsilon}{2^{n-1}}}{k}$, where $k = (1-\varepsilon) + \frac{n\varepsilon}{2^{n-1}}$.
- $\Phi(\nu), \Psi(\nu) \in \mathcal{M}(\nu)$.
- $\Phi(\nu) = P_0 \iff \Psi(\nu) = P_0 \iff P_0$ uniform.

Pari mutuel models

Given a probability measure P_0 and a distorsion value $\delta > 0$, the pari-mutuel model they induce is

$$\nu(A) = \max\{(1+\delta)P_0(A) - \delta, 0\}.$$

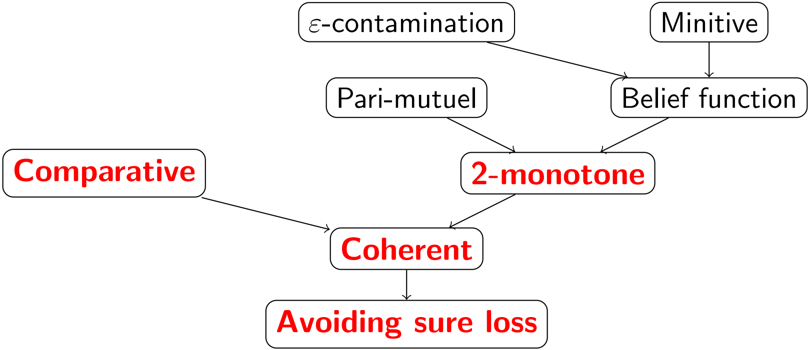
It is 2-monotone, but not necessarily a belief function.

If $\delta < \min_i \frac{P_0(\{i\})}{1-P_0(\{i\})}$, then:

- $\Phi(\nu)(i) = (1+\delta)P_0(\{i\}) - \frac{\delta}{n}\ \forall i$.
- $\Psi(\nu)(i) = \frac{(1+\delta)P_0(\{i\}) - \frac{\delta}{2^{n-1}}}{k}$, where $k = (1+\delta) - \frac{n\delta}{2^{n-1}}$.
- $\Phi(\nu), \Psi(\nu) \in \mathcal{M}(\nu)$.
- $\Phi(\nu) = P_0 \iff \Psi(\nu) = P_0 \iff P_0$ uniform.

More general models

We consider next models that are not 2-monotone in general:



Coherent lower probabilities

ν is coherent $\iff$ $\nu(A) = \min\{P(A) : P \in \mathcal{M}(\nu)\}\ \forall A$. They correspond to exact games.

- $\nu = \min\{P_1, P_2\} \Rightarrow \Phi(\nu) = \Psi(\nu) = (P_1 + P_2)/2$.
- $|\Omega| = 4 \Rightarrow \Phi(\nu) \in \mathcal{M}(\nu)$, but $\Psi(\nu)$ need not belong to $\mathcal{M}(\nu)$.
- $|\Omega| = 5 \not\Rightarrow \Phi(\nu) \in \mathcal{M}(\nu)$ (Baroni and Vicig).

Comparative probabilities

Let $\mathcal{I} \subseteq \Omega \times \Omega$. The (elementary) comparative model determined by $\mathcal{I}$ is the lower envelope $\underline{P}$ of the set

$$\mathcal{M} := \{P \text{ probability measure} : P(\{i\}) \geq P(\{j\})\ \forall (i,j) \in \mathcal{I}\}.$$

It determines a coherent lower probability, but not necessarily a 2-mononote one.

- $\Phi(\nu), \Psi(\nu) \in \mathcal{M}(\nu)$.
- $\Phi(\nu)$ need not coincide with the center of gravity of $\mathcal{M}(\nu)$.

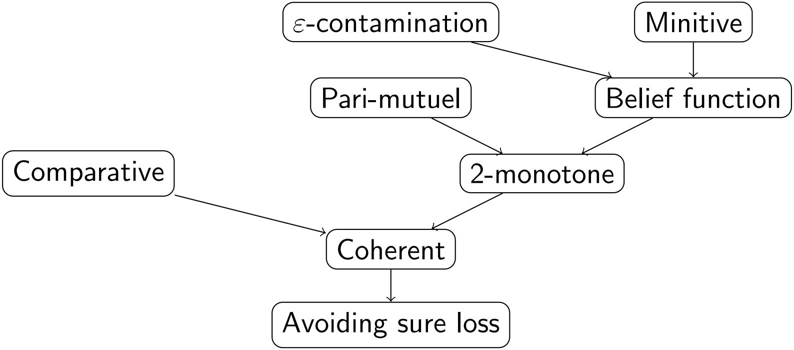
Lower probabilities avoiding sure loss

We say that ν avoids sure loss when $\mathcal{M}(\nu) \neq \emptyset$. They correspond to balanced games.

If ν is coherent, then it avoids sure loss, but the converse is not true.

- ν avoids sure loss $\not\Rightarrow \Phi(\nu), \Psi(\nu) \in \mathcal{M}(\nu)$, not even with $|\mathcal{X}| = 3$.

Relationships between the IP models



Summary of the results

ν	$\Phi(\nu)$ $\in \mathcal{M}(\nu)$?	$\Psi(\nu)$ $\in \mathcal{M}(\nu)$?	$\Phi(\nu)$ center of $\mathcal{M}(\nu)$?	$\sum_i B(\nu)(i) \leq 1$?
ASL	NO	NO	NO	NO
Coherent, $\lvert\Omega\rvert \leq 3$	YES	YES	YES	NO
Coherent, $\lvert\Omega\rvert \leq 4$	YES	NO	NO	NO
Comparative	YES	YES	NO	NO
Coherent	NO	NO	NO	NO
Pari-mutuel	YES	YES	YES	NO
2-monotone	YES	NO	YES	NO
Belief	YES	NO	YES	YES
Minitive	YES	YES	YES	YES

Open problems

- Study of other IP models: probability intervals, *p*-boxes.
- Comparison with other probability transformations.
- Do Shapley and Banzhaf satisfy desirable properties of probability transformations?
- Traslation of desirable properties of game solutions into the context of imprecise probabilities.

References

- P. Baroni, P. Vicig, *An uncertainty interchange format with imprecise probabilities*. Int. J. of Approximate Reasoning, 2005.
- D. Dubois, H. Prade, *Unfair coins and necessity measures: towards a possibilistic interpretation of histograms*. Fuzzy Sets and Systems, 1983.
- P. Smets, *Decision making in the TBM: the necessity of the pignistic transformation*. Int. J. of Approximate Reasoning, 2005.
- L. Shapley, *Cores of convex games*. Int. J. of Game Theory, 1971.

Thank you for your attention!

More information about our work at:

http://unimode.grupos.uniovi.es/

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