A study of the Pari-Mutuel Model from the point of view of Imprecise Probabilities

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Overview

The Pari-Mutuel Model

The Pari-Mutuel Model...

- Distortion model.
- Originated in horse racing.
- Applied in finance, risk analysis, ...

Precise probability P_0

Taxation from the house $\delta > 0$

$$\underline{P}(A) = \max\{(1+\delta)P_0(A) - \delta, 0\}$$
$$\overline{P}(A) = \min\{(1+\delta)P_0(A), 1\}$$

$$\downarrow$$

$$\mathcal{M}(P_0, \delta) = \{P \text{ prob.} \mid \underline{P}(A) \leq P(A) \leq \overline{P}(A)\}$$

The Pari-Mutuel Model

What is known about the Pari-Mutuel Model?

- Statistical Reasoning with Imprecise Probabilities, P. Walley. Chapman and Hall, London, 1991.
- Inference and risk measurement with the pari-mutuel model, R. Pelessoni et al. IJAR, 2010.
- A framework for imprecise robust one-class classification models, L. Utkin. Int. J. Mach. Learn. & Cyber., 2014.

The Pari-Mutuel Model

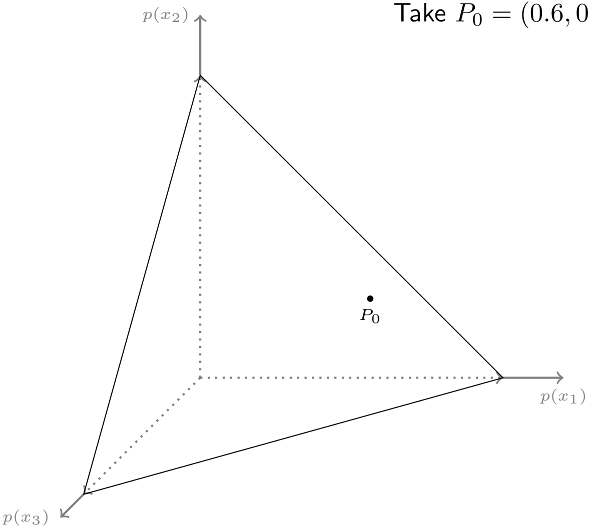
Pari-Mutuel Model

Connection between the PMM and other models

Number of extreme points induced by a PMM

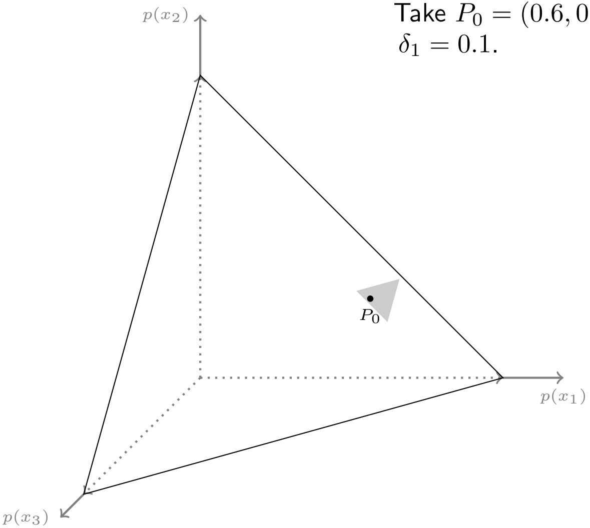
Information fusion with PMMs

Conclusions



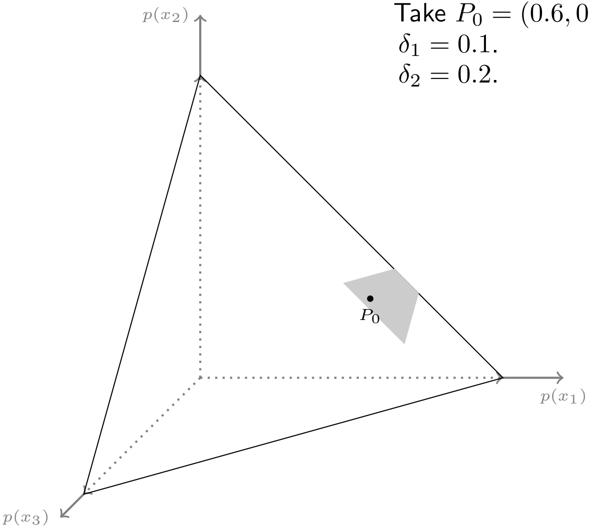
Take $P_0 = (0.6, 0.3, 0.1)$.

The Pari-Mutuel Model



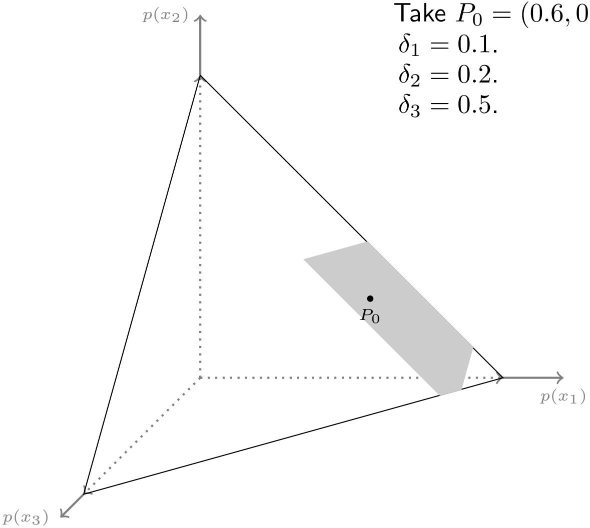
Take $P_0 = (0.6, 0.3, 0.1)$.
$\delta_1 = 0.1$.

The Pari-Mutuel Model



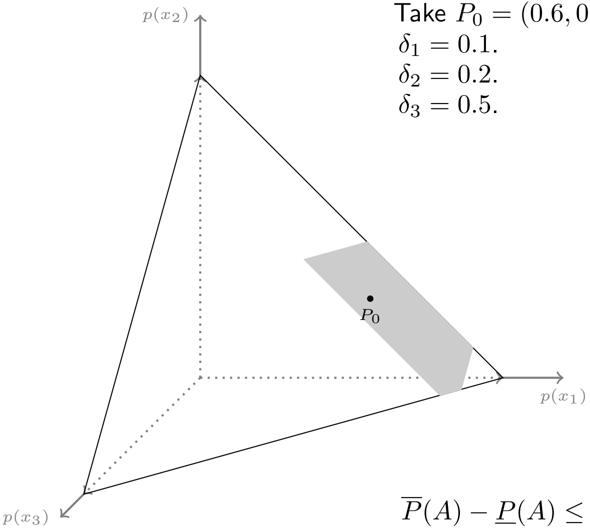
Take $P_0 = (0.6, 0.3, 0.1)$.
$\delta_1 = 0.1$.
$\delta_2 = 0.2$.

The Pari-Mutuel Model



Take $P_0 = (0.6, 0.3, 0.1)$.
$\delta_1 = 0.1$.
$\delta_2 = 0.2$.
$\delta_3 = 0.5$.

The Pari-Mutuel Model



Take $P_0 = (0.6, 0.3, 0.1)$.
$\delta_1 = 0.1$.
$\delta_2 = 0.2$.
$\delta_3 = 0.5$.

$$\overline{P}(A) - \underline{P}(A) \leq \delta$$
$$\overline{P}(A), \underline{P}(A) \in (0,1) \Rightarrow \overline{P}(A) - \underline{P}(A) = \delta$$

Overview

Probability intervals

$$\mathcal{I} = \{[l_i, u_i] \mid i = 1, \ldots, n\}.$$

$$\mathcal{M}(\mathcal{I}) = \{P \text{ prob.} \mid l_i \leq P(\{x_i\}) \leq u_i\}.$$

l, u are the lower and upper envelopes of $\mathcal{M}(\mathcal{I})$.

Theorem

Let P_0 be a probability, $\delta > 0$ and $\underline{P}, \overline{P}$ the lower and upper probability induced by the PMM. Define the probability interval:

$$\mathcal{I} = \{[\underline{P}(\{x_i\}), \overline{P}(\{x_i\})] \mid i = 1, \ldots, n\}.$$

Then $\mathcal{M}(\mathcal{I}) = \mathcal{M}(P_0, \delta)$, or equivalently $\underline{P} = l$ and $\overline{P} = u$.

Belief functions

Define $k = \min\{|A| : \underline{P}(A) > 0\}$.

Theorem

Let $\underline{P}$ be the lower probability induced by a PMM (P_0, δ). $\underline{P}$ is a belief function if and only if one of the following conditions hold:

1. $k = n$.

Focal sets: X, $m(X) = 1$.

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1. $k = n$.
2. $k = n - 1$ *and* $\sum_{i=1}^{n} \underline{P}(X \setminus \{x_i\}) \leq 1$.

Focal sets: X, $X \setminus \{x\}\ \forall x \in X$.

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1. $k = n$.
2. $k = n - 1$ *and* $\sum_{i=1}^{n} \underline{P}(X \setminus \{x_i\}) \leq 1$.
3. $k < n - 1$, $\exists! B$ *with* $|B| = k$ *and* $\underline{P}(B) > 0$, *and* $\underline{P}(A) > 0$ *if and only if* $B \subseteq A$.

Focal sets: B, $B \cup \{x\}$, $\forall x \notin B$.

Belief functions

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3. $k < n - 1$, $\exists! B$ *with* $|B| = k$ *and* $\underline{P}(B) > 0$, *and* $\underline{P}(A) > 0$ *if and only if* $B \subseteq A$.
4. $k < n - 1$, $\exists! B$ *with* $|B| = k - 1$ *and* $\delta = \frac{P_0(B)}{1 - P_0(B)}$, *and* $\underline{P}(A) > 0$ *if and only if* $B \subset A$.

Focal sets: $B \cup \{x\}$, $\forall x \notin B$.

Overview

Extreme points induced by a PMM

Since $\underline{P}$ is 2-monotone, all extreme points P_σ are generated by the permutations σ of $\{1,\ldots,n\}$:

$$P_\sigma(\{x_{\sigma(1)}\}) = \overline{P}(\{x_{\sigma(1)}\}).$$
$$P_\sigma(\{x_{\sigma(1)},\ldots,x_{\sigma(k)}\}) = \overline{P}(\{x_{\sigma(1)},\ldots,x_{\sigma(k)}\}).$$

Proposition

Let $\underline{P},\overline{P}$ be the lower and upper probability induced by a PMM (P_0,δ). P_σ is given by:

$$P_\sigma(\{x_{\sigma(i)}\}) = \overline{P}(\{x_{\sigma(i)}\}) \quad \forall i = 1,\ldots,j-1.$$
$$P_\sigma(\{x_{\sigma(j)}\}) = \underline{P}(\{x_{\sigma(j)},\ldots,x_{\sigma(n)}\}),$$
$$P_\sigma(\{x_{\sigma(j+1)}\}) = \ldots = P_\sigma(\{x_{\sigma(n)}\}) = 0,$$

where j satisfies

$$\overline{P}(\{x_{\sigma(1)},\ldots,x_{\sigma(j-1)}\}) < \overline{P}(\{x_{\sigma(1)},\ldots,x_{\sigma(j)}\}) = 1.$$

Maximal number of extreme points

Theorem

Then maximal number of extreme points of $\mathcal{M}(P_0, \delta)$ is:

1. $\frac{n}{2}\binom{n}{\frac{n}{2}}$, *if n is even.*
2. $\frac{n+1}{2}\binom{n}{\frac{n+1}{2}}$, *if n is odd.*

- It coincides with the maximal number of extreme points induced by a probability interval.
- The upper bound can be attained for the uniform distribution.

Number of extreme points

Given the PMM (P_0, δ) inducing $\underline{P}, \overline{P}$, define:

$$\mathcal{L} = \{A \subseteq X \mid \overline{P}(A) = 1\}.$$

Proposition

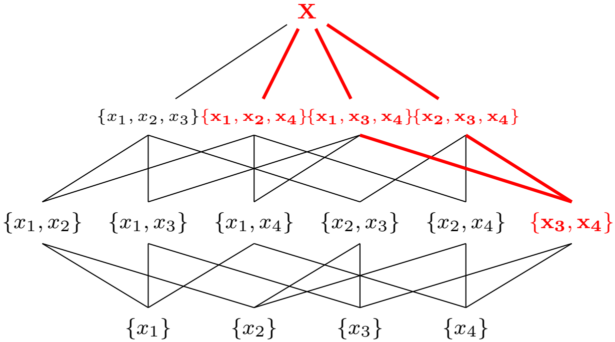
Given a PMM (P_0, δ), the number of extreme points of $\mathcal{M}(P_0, \delta)$ is bounded above by:

$$\sum_{A \in \mathcal{L}} \left| \bigcap_{B \subseteq A, B \in \mathcal{L}} B \right|.$$

Furthermore, the upper bound is attained if and only if $P_0(A) > \frac{1}{1+\delta}$ for any $A \in \mathcal{L}$.

Number of extreme points

$X = \{x_1, x_2, x_3, x_4\}$, $P_0 = (0.05, 0.15, 0.2, 0.6)$, $\delta = 0.3$:



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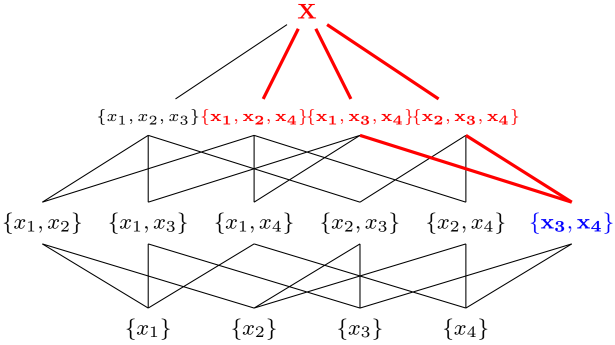

$$\{x_3, x_4\} \longrightarrow \left| \bigcap_{B \subseteq \{x_3,x_4\}, B \in \mathcal{L}} B \right| = |\{x_3, x_4\}| = 2.$$

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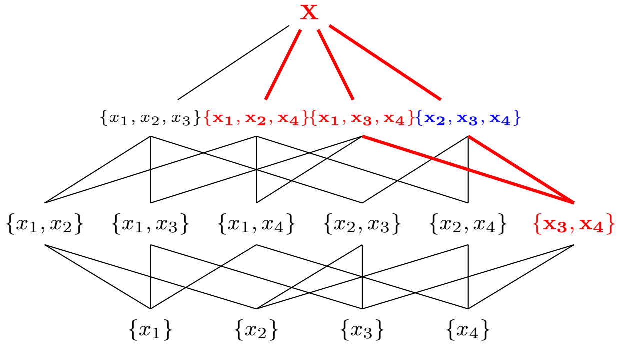

$$\{x_2, x_3, x_4\} \longrightarrow \left| \bigcap_{B \subseteq \{x_2, x_3, x_4\}, B \in \mathcal{L}} B \right| = |\{x_3, x_4\}| = 2.$$

Number of extreme points

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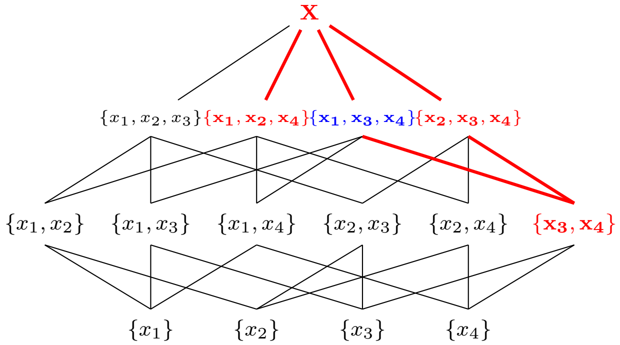

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Number of extreme points

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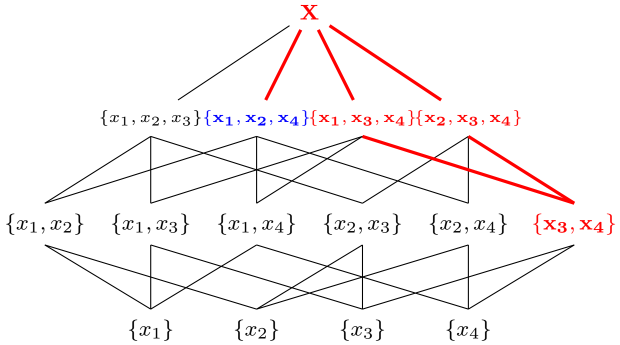
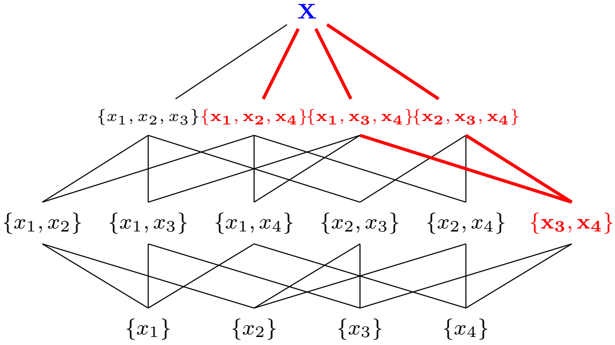


$$\{x_1, x_2, x_4\} \longrightarrow \left| \bigcap_{B \subseteq \{x_1, x_2, x_4\}, B \in \mathcal{L}} B \right| = |\{x_1, x_2, x_4\}| = 3.$$

Number of extreme points

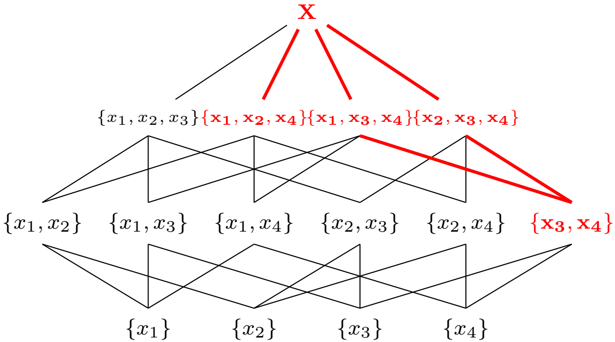
$X = \{x_1, x_2, x_3, x_4\}$, $P_0 = (0.05, 0.15, 0.2, 0.6)$, $\delta = 0.3$:



$$X \longrightarrow \left| \bigcap_{B \subseteq X, B \in \mathcal{L}} B \right| = |\{x_4\}| = 1.$$

Number of extreme points

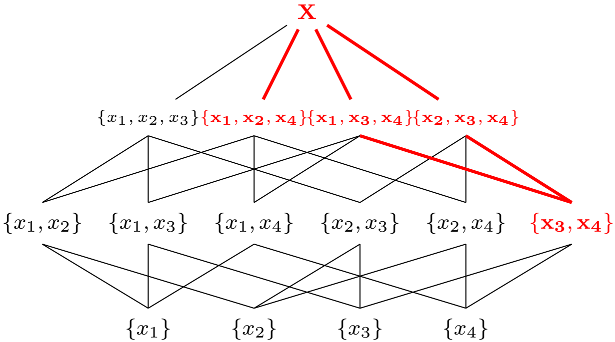
$X = \{x_1, x_2, x_3, x_4\}$, $P_0 = (0.05, 0.15, 0.2, 0.6)$, $\delta = 0.3$:



$$|ext(\mathcal{M}(P_0, \delta))| \leq 2 + 2 + 2 + 3 + 1.$$

Number of extreme points

$$X=\{x_1,x_2,x_3,x_4\},\ P_0=(0.05,0.15,0.2,0.6),\ \delta=0.3:$$



$$|ext(\mathcal{M}(P_0,\delta))|=2+2+2+3+1.$$

Overview

Combining multiple PMMs

Given two PMMs $(P_0^1, \delta_1), (P_0^2, \delta_2)$, we study:

- Conjunction: $\mathcal{M}(P_0^1, \delta_1) \cap \mathcal{M}(P_0^2, \delta_2)$.
- Disjunction: $\mathcal{M}(P_0^1, \delta_1) \cup \mathcal{M}(P_0^2, \delta_2)$.
- Mixture: $\varepsilon\mathcal{M}(P_0^1, \delta_1) + (1-\varepsilon)\mathcal{M}(P_0^2, \delta_2)$.

Conjunction

Pari-Mutuel Model

Connection between the PMM and other models

Number of extreme points induced by a PMM

Information fusion with PMMs

Conclusions

Proposition

$\mathcal{M}(P_0^1, \delta_1) \cap \mathcal{M}(P_0^2, \delta_2)$ *is non-empty if and only if:*

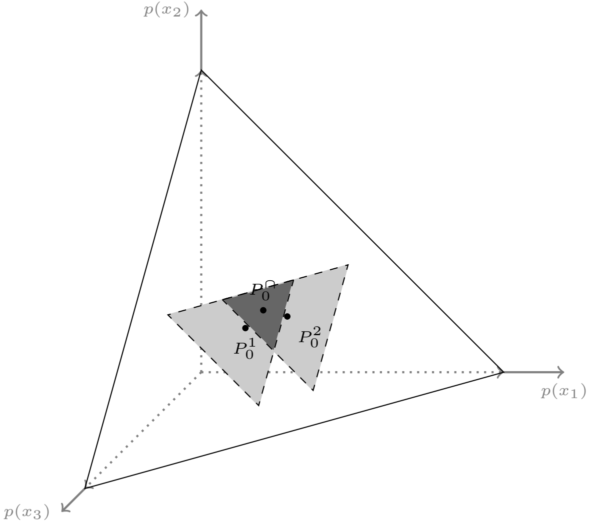
$$\sum_{x \in X} \min \left\{(1+\delta_1)P_0^1(\{x\}), (1+\delta_2)P_0^1(\{x\}), 1\right\} \geq 1.$$

Then, it is induced by a PMM $(P_0^{\cap}, \delta^{\cap})$ *given by:*

$$\delta^{\cap} = \left(\sum_{x \in X} \min \left\{(1+\delta_1)P_0^1(\{x\}), (1+\delta_2)P_0^1(\{x\}), 1\right\}\right) - 1.$$

$$P_0^{\cap} = \frac{\min \left\{(1+\delta_1)P_0^1(\{x\}), (1+\delta_2)P_0^1(\{x\})\right\}}{1+\delta^{\cap}}.$$

Conjunction



$$\mathcal{M}(P_0^1, \delta_1) \cap \mathcal{M}(P_0^2, \delta_2)$$

Disjunction

Pari-Mutuel Model

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Conclusions

Proposition

- *Neither $\mathcal{M}(P_0^1,\delta_1)\cup\mathcal{M}(P_0^2,\delta_2)$ nor its convex hull are induced by a PMM.*
- *However, they can be outer-approximated by a PMM:*

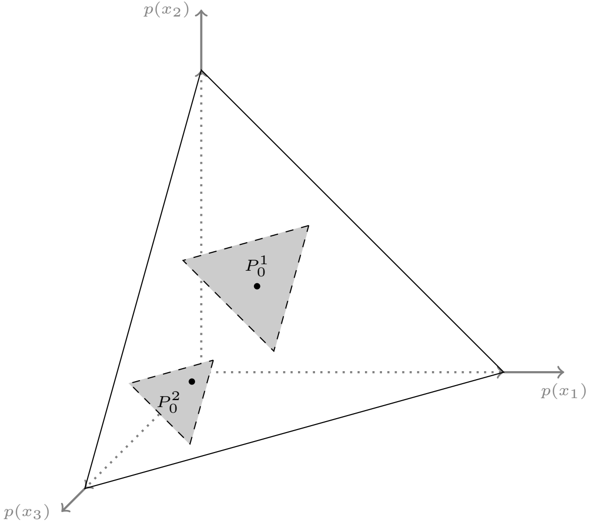
$$conv\left(\mathcal{M}(P_0^1,\delta_1)\cup\mathcal{M}(P_0^2,\delta_2)\right)\subseteq\mathcal{M}(P_0^{\cup},\delta^{\cup}),$$

given by:

$$\delta^{\cup}=\left(\sum_{x\in X}\max\left\{(1+\delta_1)P_0^1(\{x\}),(1+\delta_2)P_0^2(\{x\})\right\}\right)-1.$$

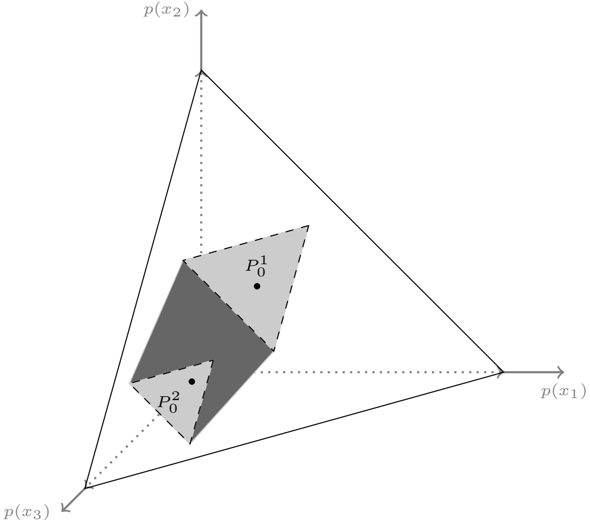
$$P_0^{\cup}=\frac{\max\left\{(1+\delta_1)P_0^1(\{x\}),(1+\delta_2)P_0^2(\{x\})\right\}}{1+\delta^{\cup}}.$$

Disjunction



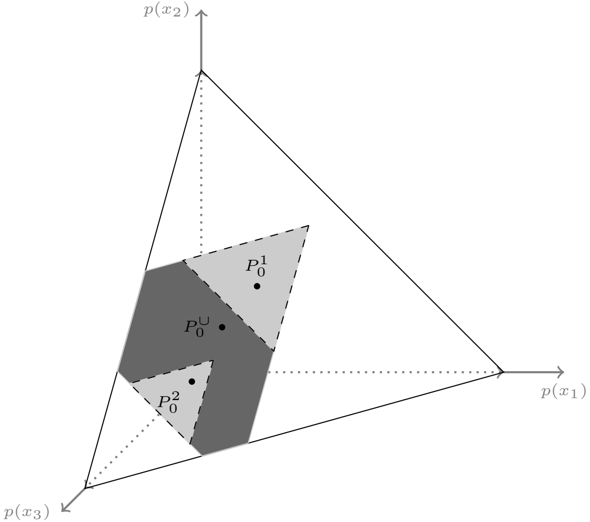
$$\mathcal{M}(P_0^1, \delta_1) \cup \mathcal{M}(P_0^2, \delta_2)$$

Disjunction



$$conv(\mathcal{M}(P_0^1, \delta_1) \cup \mathcal{M}(P_0^2, \delta_2))$$

Disjunction



$$\mathcal{M}(P_0^{\cup}, \delta^{\cup})$$

Mixture

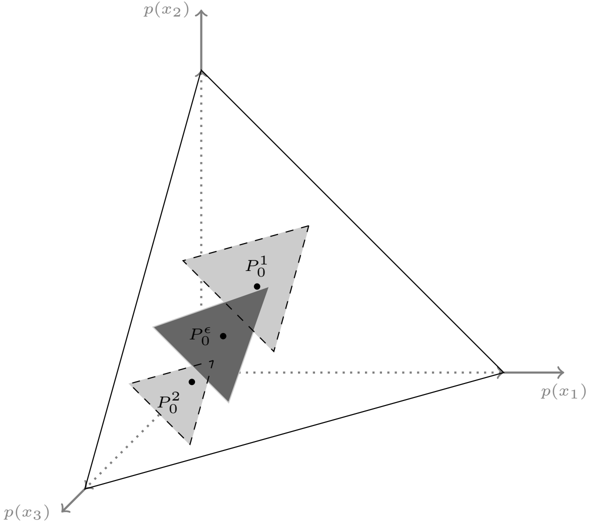
Proposition

$\varepsilon\mathcal{M}(P_0^1, \delta_1) + (1-\varepsilon)\mathcal{M}(P_0^2, \delta_2)$ is induced by a PMM $(P_0^\varepsilon, \delta^\varepsilon)$ given by:

$$\delta^\varepsilon = \varepsilon(1+\delta_1) + (1-\varepsilon)(1+\delta_2) - 1.$$

$$P_0^\varepsilon = \frac{\varepsilon(1+\delta_1)P_0^1(\{x\}) + (1-\varepsilon)(1+\delta_2)P_0^2(\{x\})}{1+\delta^\varepsilon}.$$

Mixture



$$\epsilon\mathcal{M}(P_0^1, \delta_1) + (1-\epsilon)\mathcal{M}(P_0^2, \delta_2)$$

Overview

Conclusions

The PMM as an imprecise probability model:

- Extension to gambles (*Pelessoni et al.*, *Walley*).
- The PMM and risk measures (*Pelessoni et al.*).
- Conditioning a PMM (*Pelessoni et al.*).
- PMM with a uniform distribution (*Utkin*).

Conclusions

The PMM as an imprecise probability model:

- Extension to gambles (*Pelessoni et al.*, *Walley*).
- The PMM and risk measures (*Pelessoni et al.*).
- Conditioning a PMM (*Pelessoni et al.*).
- PMM with a uniform distribution (*Utkin*).
- Connection with other models of the IP Theory.
- Extreme points of $\mathcal{M}(P_0, \delta)$.
- Merging information given in terms of PMMs.

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