Monotonicity in Bayesian Networks for Computerized Adaptive Testing

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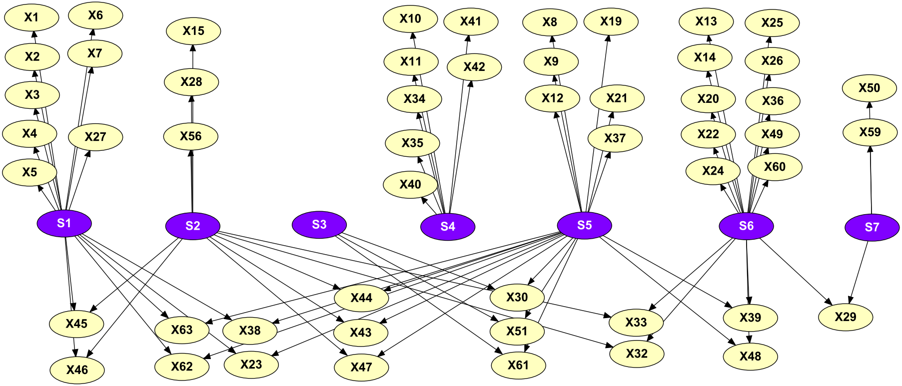
Adaptive Testing

- Selection of questions' subsets.
- Shorter test versions.
- Individual sets of questions.
- Improved precision and understanding of student's skills.
- Students are modeled by a student model. BN in our case.

Adaptive Testing Procedure

- Select a next question.
- Ask the question.
- Update the model.
- Estimate student's skills/answers.

Expert Network Model



Single Parent Monotonicity

With a natural ordering of states of a skill variable S_j

$$s_{j,1} \prec \ldots \prec s_{j,m_j} \quad ,$$

the monotonic effect on its child question variable X_i is

$$s_{j,k} \preceq s_{j,l} \quad \Rightarrow \quad P(X_i = 1 | S_j = s_{j,k}, \boldsymbol{s}) \; \leq \; P(X_i = 1 | S_j = s_{j,l}, \boldsymbol{s}) \quad .$$

Model Monotonicity

With multiple parents of a question X_i and their states configurations $(\boldsymbol{s}^i, \boldsymbol{r}^i)$,
we create a partial ordering of these configurations based on their effect on the child X_i

$$\boldsymbol{s}^i \quad \preceq_i \quad \boldsymbol{r}^i \quad .$$

Then the monotonicity condition is

$$\boldsymbol{s}^i \preceq_i \boldsymbol{r}^i \quad \Rightarrow \quad P(X_i = 1 | \boldsymbol{S}^i = \boldsymbol{s}^i) \ \leq \ P(X_i = 1 | \boldsymbol{S}^i = \boldsymbol{r}^i) \quad .$$

Reasons to Use Monotonicity

- Sensible requirement in many applications.
- Experts acceptance.
- Additional information $\rightarrow$ easier/more precise learning.

Learning Parameters under Monotonicity

- Monotonicity:
  van der Gaag, L., Bodlaender, H. L., and Feelders, A. J. (2004). Monotonicity in Bayesian networks. UAI2004
- Gradient learning method (motivation method):
  Altendorf, E. E., Restificar, A. C., and Dietterich, T. G. (2005). Learning from Sparse Data by Exploiting Monotonicity Constraints. UAI2005
- Isotonic regression EM (comparison method):
  Masegosa, A. R., Feelders, A. J., and van der Gaag, L. (2016). Learning from in- complete data in Bayesian networks with qualitative influences. IJAR

Learning Parameters under Monotonicity

Given the model parameters $\boldsymbol{\mu} = (\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_n, \boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_m)$,

$$\theta_{i,\boldsymbol{s}^i} = P(X_i = 0 | \boldsymbol{S}^i = \boldsymbol{s}^i), \; \boldsymbol{\theta}_i = (\theta_{i,\boldsymbol{s}^i})_{\boldsymbol{s}^i \in Val(\boldsymbol{S}^i)} \;,$$

we use the model's log likelihood

$$LL(\boldsymbol{\mu}) \;,$$

which we penalize:

$$\begin{aligned} p(\theta_{i,\boldsymbol{s}^i}, \theta_{i,\boldsymbol{r}^i}) &= exp(c \cdot (\theta_{i,\boldsymbol{r}^i} - \theta_{i,\boldsymbol{s}^i})) \\ LL'(\boldsymbol{\mu}, c) &= LL(\boldsymbol{\mu}) - \sum_{i \in \boldsymbol{N}} \sum_{\boldsymbol{s}^i \preceq_i \boldsymbol{r}^i} p(\theta_{i,\boldsymbol{s}^i}, \theta_{i,\boldsymbol{r}^i}) \;. \end{aligned}$$

Learning Parameters under Monotonicity

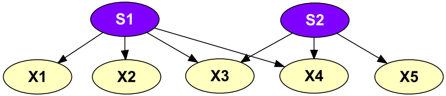
Penalized log likelihood

$$LL'(\boldsymbol{\mu}, c) = LL(\boldsymbol{\mu}) - \sum_{i \in \boldsymbol{N}} \sum_{\boldsymbol{s}^i \preceq_i \boldsymbol{r}^i} p(\theta_{i,\boldsymbol{s}^i}, \theta_{i,\boldsymbol{r}^i})$$

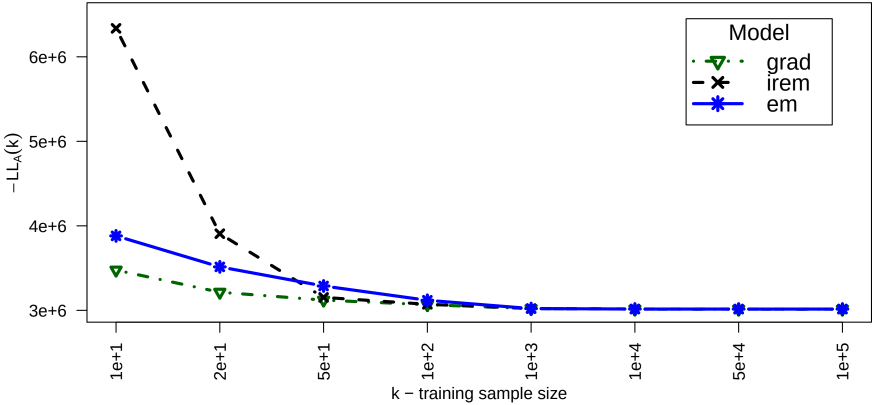
is optimized using gradient methods.

Experiments

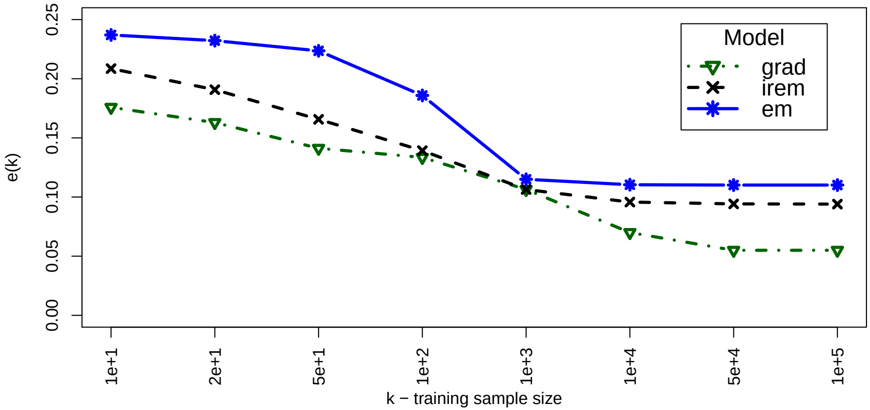
- Experimental evaluation with
  - Empirical data set - Math test, expert model, 281 cases
  - Synthetic data set - 100 000 cases



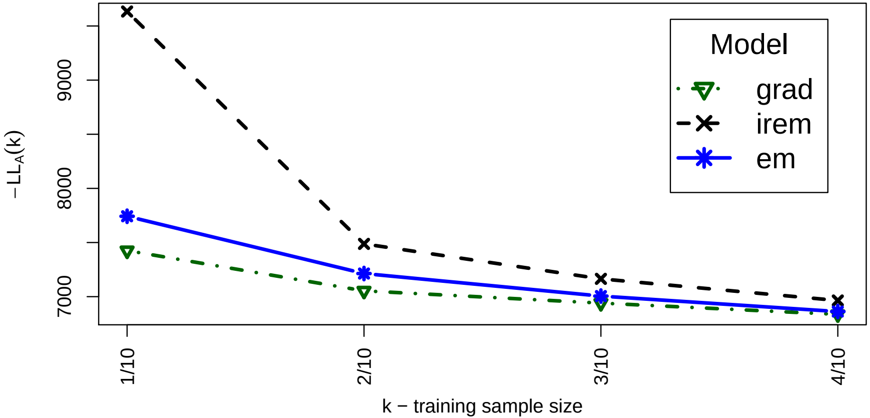
Results - Synthetic Model Log Likelihood



Results - Synthetic Model Parameters



Results - Empirical Model Log Likelihood



Conclusions

- Gradient method for monotonic parameters learning with hidden variables.
- Provides good results for small training sets.
- Comparable results with other methods for larger training sets.
- Generalization for less specific network structure is required.