Analogical inequalities

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Moving from equality to inequality

- Analogical proportion:

  "*a* is to *b* **as** *c* is to *d*"

- Analogical *inequality*

  "*a* is to *b* **at least as much as** *c* is to *d*"

Contents

Law, Thome, Cord. Quadruplet-wise image similarity learning (ICCV'2013)

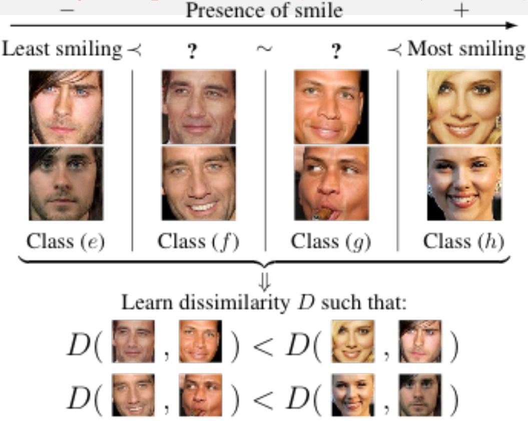

Figure 1. Quadruplet-wise (Qwise) strategy on 4 face classes ranked according to the degree of presence of smile. Instead of working on pairwise relations that present some flaws (see text), Qwise strategy defines quadruplet-wise constraints to express that dissimilarities between examples from (f) and (g) should be smaller than dissimilarities between examples from (e) and (h).

Preference statements as comparisons of differences

comparison of pairs

$$x_{-i}\alpha \preceq y_{-i}\beta$$

the “difference” between α and β
on criterion i
is smaller than (i.e., does not compensate)
the “difference” between the vectors x_{-i} and y_{-i}
on the rest of the criteria

sounds like analogical inequalities

M. Pirlot, H. Prade, G. Richard. Completing preferences by means of analogical proportions. Proc. 13th Int. Conf. on Modeling Decisions for Artificial Intelligence (MDAI’16), Andorra, Sept. 2016.

Analogical proportions

- "*a* is to *b* as *c* is to *d*"
  a differs from *b* as *c differs* from *d*
  and *b differs* from *a* as *d differs* from *c*".
- $a : b :: c : d \triangleq$
  $((a \wedge \neg b) \equiv (c \wedge \neg d)) \wedge ((\neg a \wedge b) \equiv (\neg c \wedge d))$
- it uses dissimilarity indicators only

Analogical proportion truth table

Boolean patterns making analogical proportion true

a	b	c	d
0	0	0	0
1	1	1	1
0	0	1	1
1	1	0	0
0	1	0	1
1	0	1	0

code independent property:

$$a : b :: c : d \Leftrightarrow \neg a : \neg b :: \neg c : \neg d$$

Graded analogical proportion

- Attributes not necessarily Boolean: graded extensions of logical proportions of interest
- analogical proportion : 2 options that make sense

$$a : b ::_L c : d = \begin{cases} 1 - \mid (a-b) - (c-d) \mid, \\ \quad \text{if } a \geq b \text{ and } c \geq d, \text{ or } a \leq b \text{ and } c \leq d \\ 1 - \max(\mid a-b \mid, \mid c-d \mid), \\ \quad \text{if } a \leq b \text{ and } c \geq d, \text{ or } a \geq b \text{ and } c \leq d \end{cases}$$

- Coincides with $a : b :: c : d$ on $\{0, 1\}$
- Equal to **1** if and only if $(a-b) = (c-d)$
- $a : b ::_L c : d = \mathbf{0}$ when the change inside one of (a, b) or (c, d) is *maximal*, while the other pair shows either no change, or an *opposite* change

Boolean analogical inequalities

"*a* is to *b* **at least as much as** *c* is to *d*"

$$a : b \ll c : d =$$
$$((a \wedge \neg b) \rightarrow (c \wedge \neg d)) \wedge ((\neg a \wedge b) \rightarrow (\neg c \wedge d))$$

a	b	c	d	$a : b \ll c : d$	a	b	c	d	$a : b \ll c : d$
0	0	0	0	**1**	1	0	0	0	0
0	0	0	1	**1**	1	0	0	1	0
0	0	1	0	**1**	1	0	1	0	**1**
0	0	1	1	**1**	1	0	1	1	0
0	1	0	0	0	1	1	0	0	**1**
0	1	0	1	**1**	1	1	0	1	**1**
0	1	1	0	0	1	1	1	0	**1**
0	1	1	1	0	1	1	1	1	**1**

Properties

$a : b \ll c : d$ holds true for the 6 patterns that makes analogical proportion true,
plus the 4 patterns 0001, 0010, 1110, 1101
$a : b \ll c : d$ true iff $(a : b :: c : d) \vee (a \equiv b)$ true

- $a : b \ll a : b$
- $a : b :: c : d \Rightarrow a : b \ll c : d$
- $a : b :: c : d \Leftrightarrow ((a : b \ll c : d) \wedge (c : d \ll a : b))$
- $(a : b \ll c : d) \Leftrightarrow (\neg a : \neg b \ll \neg c : \neg d)$

Graded analogical inequalities

- keeping min for extending the central $\wedge$
  $1 - \mid s - t \mid$ for $\equiv$
  for the 4 expressions of the form $s \wedge \neg t$, the bounded difference $\max(0, s - t)$
- $a : b \ll c : d =$
  $$\begin{cases} \min(1, 1 - ((b-a) - (d-c)) & \text{if } a \leq b \text{ and } c \leq d \\ \min(1, 1 - ((a-b) - (c-d)) & \text{if } a \geq b \text{ and } c \geq d \\ 1 - (b-a) & \text{if } a \leq b \text{ and } c \geq d \\ 1 - (a-b) & \text{if } a \geq b \text{ and } c \leq d \end{cases}$$
- can be read
  "*c* is more different from *d* than *a* is from *b*"

Properties

- coincides with Boolean definition if $a,b,c,d \in \{0,1\}$
- $a : b \ll a : b = 1$;
- $a : b :: c : d \leq a : b \ll c : d$;
- $a : b :: c : d = \min((a : b \ll c : d), (c : d \ll a : b))$
- $(a : b \ll c : d) = ((1-a) : (1-b) \ll (1-c) : (1-d))$

In particular, $a : b \ll c : d = 1$ if and only if

- $a = b$, or
- $|b - a| \leq |d - c|$ if $a \leq b$ & $c \leq d$, or if $b \leq a$ & $d \leq c$

Moreover $a : b \ll c : d = 0$ if and only if

- $|b - a| = 1$ and $|d - c| = 0$, or
- $b - a = 1$ and $c \geq d$, or
- $a - b = 1$ and $c \leq d$

Continuous analogical inequalities

$$a : b \ll b : c = \begin{cases} \min(1, 1 + (a + c) - 2b) & \text{if } a \leq b \leq c \\ \min(1, 1 + 2b - (a + c)) & \text{if } a \geq b \geq c \\ 1 - (b - a) & \text{if } a \leq b \text{ and } b \geq c \\ 1 - (a - b) & \text{if } a \geq b \text{ and } b \leq c \end{cases}$$

$a : b \ll b : c = 1$ if and only if $a = b$, or

if $b \leq (a + c)/2$ when $a \leq b \leq c$, or

if $b \geq (a + c)/2$ when $a \geq b \geq c$

i.e., if and only if b is closer to a than to c

It means that the difference between b and c is greater or equal to the one between a and b and the differences are oriented in the same way

Concluding remarks

- can be generalized to vectors in a component-wise manner
- $a : b \ll c : d$ does not exactly amount at comparing distance values
  e.g., $a : b \ll c : d = 0$, while $\mid a - b \mid \leq \mid c - d \mid$
  with $a = d = 0$ and $b = c = 1$
  $a : b \ll c : d$ is a graded estimate of the extent to which $a - b \leq c - d$ is satisfied
- when extended to the multiple-valued case, might be of interest in *visual multiple-class categorization task*