Axiomatization of an importance index for Generalized Additive Independence models

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Introduction

MultiCriteria Decision Analysis (MCDA)

- Central question: to be able to quantify the importance of each attribute.
  - Simple models : additive models (additive utility, WAM).
  - Complex models :
    - Choquet integral: importance index = the Shapley value.
    - GAI (Fishburn 1967).

Aim

- Define an importance index for discrete GAI models.

Basic concepts

- $N = \{1, \dots, n\}, n \in \mathbb{N}$
  - in MCDA : attributes, criteria, ...
  - in cooperative game theory : players, voters, ...
- X_i : set of values of attribute i.
- $X = X_1 \times \cdots \times X_n$: set of potential alternatives.
  - Each $x \in X$ is a vector $(x_1, \dots, x_n) := (x_i, x_{-i})$.
- GAI model (Fishburn 1967)

$$U(x) = \sum_{A \in \mathcal{S}} u_A(x_A),$$

with $\mathcal{S} \subseteq 2^N \setminus \{\emptyset\}$, and $u_A : X_A \to \mathbb{R}$

Basic concepts

- A *game* is a function $v : 2^N \to \mathbb{R}, v(\emptyset) = 0$.
- A *capacity* is a game v such that :

$$S \subseteq T \text{ implies } v(S) \leq v(T).$$

- A *Multichoice game* (Hsiao and Raghavan, 1990):

$$v : L = \{0, 1, \ldots, k\}^N \to \mathbb{R}, v(0, \ldots, 0) = 0.$$

$$\mathcal{G}(L) := \{v : L \to \mathbb{R}, v(0_N) = 0\}$$

- *k-ary capacity* (Grabisch and Labreuche 2003):

$$v \in \mathcal{G}(L) \text{ monotonic : } x \leq y \Rightarrow v(x) \leq v(y).$$

- $\forall x \in L, S(x) := \{i \in N \mid x_i > 0\}, K(x) := \{i \in N \mid x_i = k_i\}$

Discrete GAI models are multichoice games

- We consider that attributes are discrete: $X_i = \{x_i^0, \ldots, x_i^k\}$.
- Any alternative $x \in X$ is mapped to L by φ

$$(x_1^{i_1}, \ldots, x_n^{i_n}) \longmapsto \varphi(x_1^{i_1}, \ldots, x_n^{i_n}) = (i_1, \ldots, i_n).$$

- Given a GAI model U with discrete attributes
  - Assumption : $U(0, \ldots, 0) = 0$.
- We define $v : L = \{0, 1, \ldots, k\}^N \longmapsto \mathbb{R}$ by

$$U(x) =: v(\varphi(x)), x \in X,$$

and let $v(z) := v(k, \ldots, k)$ when $z \in L \setminus \varphi(X)$.

- v is a multichoice game.

Aim and related work

- $\phi : \mathcal{G}(L) \to \mathbb{R}^N$ vector of importance of the attributes.
- $k = 1$: the standard solution to give an interpretation of the model in terms of importance index : the Shapley value,
- several generalizations for multichoice games exist, e.g., Hsiao and Raghavan (1990), Peters and Zank (2005), Grabisch and Lange (2007), etc.
- axioms: linearity, null, symmetry, efficiency: $\sum_{i \in N} \phi_i(v) = v(k_N)$,
- efficiency axiom has no justification in MCDA.

Aim

- to propose a value:
  - for multichoice games,
  - for non monotonone models in MCDA.

The preferences are not necessary monotone

Example: Level of comfort of humans

- $N = \{1, 2, 3\}$,
- X_1 = temperature of the air,
- X_2 = humidity of the air,
- X_3 = velocity of the air,
- v measures the comfort level.
- $\forall i \in N$, x_{-i} fixed, $\exists \widehat{\ell_i} \in L_i$,
  - v increasing in $x_i < \widehat{\ell_i}$, decreasing in $x_i > \widehat{\ell_i}$,
  - preferences are single-peaked.
- $v(x_1, x_2, x_3)$ is not monotone in its three arguments.

Example: efficiency does not make sense

Example: simplest single-peaked function

$$\delta_y(x) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{otherwise.} \end{cases}$$

- $N = \{1, 2, 3\}, k = 2, y = (2, 1, 1)$,
- attribute x_1 has a nonzero importance since $\delta_y(x_1, \cdot, \cdot)$ is non-decreasing in x_1:
  - $0 = \delta_y(0, 1, 1) < \delta_y(2, 1, 1) = 1$.
- However efficiency implies

$$\sum_{i \in N} \phi_i(\delta_y) = \delta_y(2, 2, 2) = 0.$$

Axiomatisation: Importance index

- **Linearity (L)**: ϕ is linear on $\mathcal{G}(L)$.

Proposition

Under axiom **(L)**, $\forall i \in N, \exists a_x^i \in \mathbb{R}, \forall x \in L,$

$$\phi_i(v) = \sum_{x \in L} a_x^i v(x), \forall v \in \mathcal{G}(L).$$

i is said to be *null* for $v \in \mathcal{G}(L)$ if $v(x + 1_i) = v(x), \forall x \in L, x_i < k$.

- **Nullity (N)**: If a criterion i is *null* for $v \in \mathcal{G}(L)$, $\phi_i(v) = 0$.

Proposition

Under axioms **(L)** and **(N)**, $\forall i \in N, \exists b_x^i \in \mathbb{R}, \forall x \in L, x_i < k,$

$$\phi_i(v) = \sum_{\substack{x \in L \\ x_i < k}} b_x^i \big(v(x + 1_i) - v(x)\big), \forall v \in \mathcal{G}(L).$$

Axiomatisation: Importance index

- **Symmetry (S)**: $\forall \sigma$ on N, $\phi_{\sigma(i)}(\sigma \circ v) = \phi_i(v), \forall i \in N$.

Proposition

Under axioms **(L)**, **(N)** and **(S)**, $\forall v \in \mathcal{G}(L), \forall i \in N$,

$$\phi_i(v) = \sum_{\substack{x \in L \\ x_i < k}} b_{x_i;n(x_{-i})} \big(v(x+1_i) - v(x)\big),$$

$b_{x_i;n(x_{-i})} \in \mathbb{R}, n(x_{-i}) = (n_0, \ldots, n_k)$, $n_j = |\{\ell \in N \setminus \{i\}, x_\ell = j\}|$.

- **Invariance (I)**: Let us consider $v, w \in \mathcal{G}(L)$ such that $\forall i \in N$,

$$v(x+1_i) - v(x) = w(x) - w(x-1_i), \forall x \in L, x_i \notin \{0, k\},$$

$$v(x_{-i}, 1_i) - v(x_{-i}, 0_i) = w(x_{-i}, k_i) - w(x_{-i}, k_i - 1), \forall x_{-i} \in L_{-i},$$

then $\phi_i(v) = \phi_i(w)$.

Proposition

Under axioms **(L)**, **(N)**, **(S)** and **(I)**, $\forall v \in \mathcal{G}(L), \forall i \in N$,

$$\phi_i(v) = \sum_{\substack{x \in L \\ x_i < k}} b_{n(x_{-i})} \big(v(x+1_i) - v(x)\big).$$

Axiomatisation: Importance index

Restricted Efficiency axiom

- ϕ_i : overall added value from any x to $(x + 1_i)$.
- $\sum_{i \in N} \phi_i(v)$: overall added value when increasing simultaneously the value of all attributes of one unit.

$$\forall v \in \mathcal{G}(L), \sum_{i \in N} \phi_i(v) = \sum_{\substack{x \in L \\ x_j < k}} \big(v(x+1) - v(x)\big).$$

- **RE axiom**: $\forall y \in L \setminus \{0_N\}$,
  - $\sum_{i \in N} \phi_i(\delta_y) = \delta_y(y_{-i}, k_i) - \delta_y(y_{-j}, 0_j)$, where, $i = argmax\ y$ and $j = argmin\ y$.

$$\sum_{i \in N} \phi_i(\delta_y) = \begin{cases} +1, & \text{if } k(y) \neq 0 \text{ and } s(y) = n, \\ -1, & \text{if } k(y) = 0 \text{ and } s(y) < n, \\ 0, & \text{else.} \end{cases}$$

Axiomatisation: Importance index

Theorem

Under axioms **(L)**, **(N)**, **(I)**, **(S)** and **(RE)**, $\forall v \in \mathcal{G}(L), \forall i \in N$

$$\phi_i(v) = \sum_{x_{-i} \in L_{-i}} \frac{(n - s(x_{-i}) - 1)!k(x_{-i})!}{(n + k(x_{-i}) - s(x_{-i}))!} \big(v(x_{-i}, k_i) - v(x_{-i}, 0_i)\big).$$

- Average variation along the axis of the attribute.

Interpretation of ϕ in continuous spaces

Choquet integral w.r.t. v (Grabisch and Labreuche 2003)

$$\forall z \in [0,k]^N, \mathcal{C}_v(z) = v(x) + C_{\mu_x}(z - x),$$

where, $x = \lfloor z \rfloor, \mu_x(S) = v((x+1)_S, x_{-S}) - v(t), \forall S \subseteq N.$

Lemma

For all k-ary capacity v,

$$\phi_i(v) = \int_{[0,k]^n} \frac{\partial \mathcal{C}_v}{\partial z_i}(z)\, dz, \forall i \in N,$$

$$\phi_i(v) = \sum_{x \in \{0,\ldots,k-1\}^N} \phi_i^{Sh}(\mu_x), \forall i \in N.$$

Conclusion and future work

Conclusion

- New importance index for MCDA models :
  - Average variation along the axis of the attribute.
- The value on $\times_{i \in N}\{x_i, x_{i+1}\}$ = usual Shapley value.
- ϕ_i on continuous domain is the integrated local importance.

Future works

- Use other values: Banzhaf value.
- $\phi_i(v) = \sum_{\substack{x \in L \\ x_i < k}} p_x^i |v(x + 1_i) - v(x)|, p_x^i \in \mathbb{R}.$

Thank you for your attention!