Linear core-based criterion for testing extreme exact games

Milan Studený and Václav Kratochvíl

Institute of Information Theory and Automation of the CAS
Prague, Czech Republic

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Summary of the talk

1. Introduction: (theory of) imprecise probabilities and game theory
2. Brief overview of former related literature
3. Some basic concepts
4. Description of the criterion
5. Conclusions: web platform

Introduction: coherent lower probabilities and exact games

The notions of

- a *coherent lower probability* and that of
- an induced *credal set* (of discrete probability distributions)

are traditional topics of interest in the theory of imprecise probabilities.

These notions correspond to game-theoretical concepts of

- an *exact game* and that of
- its *core* (polytope),

widely used in the context of cooperative coalition games.

The analogy is even broader:

- a lower probability avoiding sure loss corresponds to a weaker concept of a balanced game, while
- a *2-monotone lower probability* (= capacity) corresponds to a stronger concept of a *supermodular game*, named also a convex game.

Goal: testing extreme coherent lower probabilities

The set of *coherent lower probabilities* on a finite sample space N (= frame of discernment), where $n = |N| \geq 2$, is a polytope in a 2^n-dimensional real vector space.

The set of *non-negative exact games* over a finite set N (= set of players), where $n = |N| \geq 2$, is a pointed polyhedral cone.

The relation of these two sets is that

- the extreme points of the polytope of coherent lower probabilities,

known as the extreme lower probabilities, are nothing but the generators

- of the extreme rays in the cone of non-negative exact games.

We offer a method to **test whether a ray** in the cone of non-negative exact games **is extreme**, which implicitly gives a method to test the extremity of a given coherent lower probability.

The concept of a min-representation

Some effort to develop criteria to recognize the extremity of an exact game was exerted earlier by Rosenmüller (2000).

J. Rosenmüller (2000). Game Theory: Stochastics, Information, Strategies and Cooperation. Kluwer, Boston.

He offered one necessary and one sufficient condition for the extremity based on the concept of a *min-representation* of the exact game.
However, his criteria are quite special (= have a limited scope of application).

We follow that idea and propose a more general criterion coming from the so-called *standard min-representation* of any exact game given by the list of vertices of the respective *core* polytope.

The condition presented in our conference proceedings paper is always necessary for the extremity of an exact game and it is also sufficient in a certain special case.

Computing (extreme lower probabilities)

Being motivated by questions raised by Maass (2003), Quaeghebeur and de Cooman (2008) became interested in *extreme lower probabilities* and computed these in the case of small $n = |N|$.

S. Maass (2003). Continuous linear representation of coherent lower previsions. Proceedings in Informatics 18: ISIPTA'03, Carleton Scientific, 372-382.

E. Quaeghebeur and G. de Cooman (2008). Extreme lower probabilities. Fuzzy Sets and Systems 159, 2163-2175.

Antonucci and Cuzzolin (2010) considered an enlarging transformation of a credal set with a finite number of extreme points.

A. Antonucci and F. Cuzzolin (2010). Credal set approximation by lower probabilities: application to credal networks. Lecture Notes in AI 6178: IPMU 2010, Springer, 716-725.

Analogous results achieved in different contexts

It is always useful to be aware of the correspondence between concepts from different fields.

For example, Wallner (2005) confirmed a conjecture by Weichselberger that the credal set induced by a (coherent) lower probability has at most $n!$ vertices, where $n = |N|$.

A. Wallner (2005). Maximal number of vertices of polytopes defined by f-probabilities. Proceedings of ISIPTA'05, 126-139.

However, the same result was achieved earlier by Derks and Kuipers (2001) in the context of cooperative game theory.

J. Derks and J. Kuipers (2001). On the number of extreme points of the core of a transferable utility game. Chapters in Game Theory in Honour of Stef Tijs, Kluwer, 83-97.

Supermodular case / 2-monotone lower probabilities

The criterion we offer in the proceedings paper is a modification of our former criterion to recognize extreme supermodular (= convex) games. Note that the convex games correspond to 2-monotone lower probabilities.

M. Studený and T. Kroupa (2016). Core-based criterion for extreme supermodular functions. Discrete Applied Mathematics 20, 122-151.

That former criterion offers a **necessary and sufficient condition** for **the extremity of a supermodular game**.

The supermodular criterion leads to solving a *simple linear equation system* determined by certain *combinatorial structure* (*of the core*), which concept was pinpointed even earlier by Kuipers, Vermeulen and Voorneveld.

J. Kuipers, D. Vermeulen and M. Voorneveld (2010). A generalization of the Shapley-Ichiishi result. International Journal of Game Theory 39, 585-602.

A little bit of notation

N	a finite non-empty set of *variables*, $\lvert N\rvert \geq 2$, $=$ the *sample space* in imprecise probabilities $=$ the set of *players* in game theory $=$ *random variables* with conditional independence
$\mathcal{P}(N) := \{S : S \subseteq N\}$	the power set of N,
$\mathbb{R}^N$	the set of real vectors, components indexed by N, *credal sets* / *cores* are subsets of this space
$\mathbb{R}^{\mathcal{P}(N)}$	vectors with components indexed by subsets of N, *lower probabilities* / *exact games* are in this space
$\chi_S \in \mathbb{R}^N$	given a set of variables $S \subseteq N$, the zero-one indicator of the set S.

The concepts of a core and an exact game

A (cooperative) *game* is a set function $m \in \mathbb{R}^{\mathcal{P}(N)}$ with $m(\emptyset) = 0$.

Definition (core, exact game)

The *core* of a game m is a polytope in $\mathbb{R}^N$ defined by

$$C(m) \;:=\; \{\, x \in \mathbb{R}^N \,:\; \sum_{i \in N} x_i = m(N) \;\;\&\;\; \forall\, S \subseteq N \quad \sum_{i \in S} x_i \geq m(S) \,\}\,.$$

The set of its vertices ($=$ extreme points) will be denoted by $\operatorname{ext} C(m)$.
A game m is *balanced* if $C(m) \neq \emptyset$. A balanced game is called *exact* if

$$\forall\, S \subseteq N \;\; \exists\, x \in C(m) \qquad \sum_{i \in S} x_i = m(S)\,.$$

A game m is called *ℓ-standardized* if $m(S) = 0$ for any $S \subseteq N$, $|S| \leq 1$.

Note: if m is non-negative and normalized by $m(N) = 1$ then core $=$ credal set and m is exact $\Leftrightarrow$ m is a coherent lower probability.

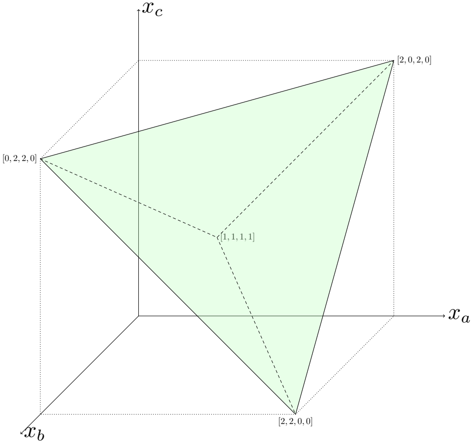
An example of the core of an exact game

$N = \{a, b, c, d\}$ $\quad m(N) = 4, \quad m(\{a, b, c\}) = 3,$

$m(\{a, b, d\}) = m(\{a, c, d\}) = m(\{b, c, d\}) = 2,$

$m(\{a, b\}) = m(\{a, c\}) = m(\{b, c\}) = 2, \quad m(S) = 0$ for other $S \subseteq N$.

$$x_d = 4 - x_a - x_b - x_c$$



	x_a	x_b	x_c	x_d
α	1	1	1	1
β	2	2	0	0
γ	2	0	2	0
δ	0	2	2	0

Extreme exact game / extreme lower probability

Let us denote the set of exact ℓ-standardized games over N by $\mathsf{E}_\ell(N)$.
It is a pointed cone in the linear space $\mathbb{R}^{\mathcal{P}(N)}$; its dimension is $2^n - n - 1$.

Definition (extreme exact game)

An ℓ-standardized exact game $m : \mathcal{P}(N) \to \mathbb{R}$ is *extreme* if it generates an extreme ray of the cone $\mathsf{E}_\ell(N)$.

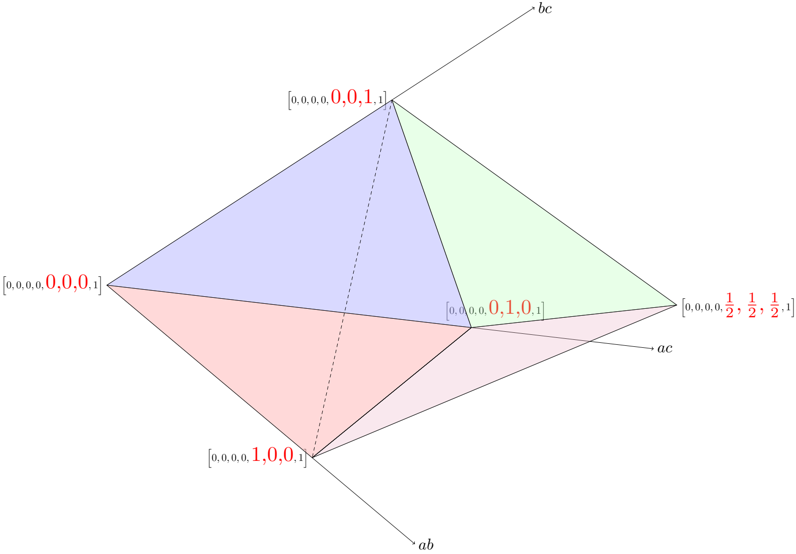
In the context of imprecise probabilities, ℓ-standardization corresponds to excluding *crisp lower probabilities* (= the credal set consists of a single probability distribution).

Every coherent lower probability can be written as a unique convex combination of its crisp and non-crisp parts. Extreme crisp lower probabilities correspond to degenerate probability distributions.

Hence, generators m of extreme rays of $\mathsf{E}_\ell(N)$, normalized by $m(N) = 1$, are nothing but *extreme non-crisp coherent lower probabilities*.

A picture for non-crisp lower probabilities, the case $|N| = 3$

Legend: $[m(\emptyset), m(\{a\}), m(\{b\}), m(\{c\}), m(\{a,b\}), m(\{a,c\}), m(\{b,c\}), m(\{a,b,c\})]$



The concept of a min-representation

Definition (min-representation of a game)

We say that $m \in \mathbb{R}^{\mathcal{P}(N)}$ has a *min-representation* (by additive functions) if a non-empty finite set $\mathcal{R} \subseteq \mathbb{R}^N$ exists such that

$$\forall\, S \subseteq N \qquad m(S) = \min_{x \in \mathcal{R}} \sum_{i \in S} x_i \,.$$

Every $x \in \mathcal{R}$ is then assigned the corresponding *tightness class* of sets

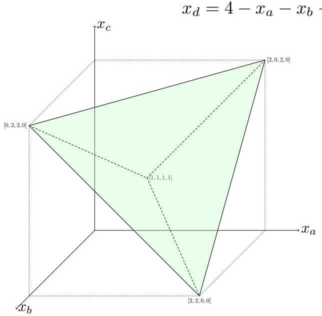
$$\mathcal{T}_x \;:=\; \{\, S \subseteq N \,:\; m(S) = \sum_{i \in S} x_i \,\}\,.$$

We say that a min-representation $\mathcal{R} \subseteq \mathbb{R}^N$ of a game m is *feasible* if, for any $x \in \mathcal{R}$, $\sum_{i \in N} x_i = m(N)$.

Thus, feasibility of a min-representation means it consists of *elements* of the core.

Example: min-representation with vertices of the core

$$N = \{a,b,c,d\} \qquad m(N) = 4, \qquad m(\{a,b,c\}) = 3,$$
$$m(\{a,b,d\}) = m(\{a,c,d\}) = m(\{b,c,d\}) = 2,$$
$$m(\{a,b\}) = m(\{a,c\}) = m(\{b,c\}) = 2, \qquad m(S) = 0 \text{ for other } S \subseteq N.$$



	x_a	x_b	x_c	x_d
α	1	1	1	1
β	2	2	0	0
γ	2	0	2	0
δ	0	2	2	0

$S = \{a,b\}$
$\min_{\alpha,\ldots,\delta} (x_a + x_b) = 2 = m(\{a,b\})$

$S = \{c,d\}$
$\min_{\alpha,\ldots,\delta} (x_c + x_d) = 0 = m(\{c,d\})$

$$\mathcal{T}_\alpha \equiv \{\, S \subseteq N \,:\, m(S) = \textstyle\sum_{i\in S} x_i^\alpha \,\} \;=\; \{\, \emptyset, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, N \,\}$$

Exactness in terms of min-representations

Proposition (min-representations for exact games)

A game $m \in \mathbb{R}^{\mathcal{P}(N)}$ is exact iff it admits a feasible min-representation $\mathcal{R}$.

Every exact game has a kind of a *standard min-representation* $\overline{\mathcal{R}}$ given by the complete list of vertices of its core: $\overline{\mathcal{R}} = \text{ext}\, C(m)$.

This standard min-representation of m is feasible and satisfies

- the linear hull of $\{\, \chi_S \,:\, S \in \mathcal{T}_x \,\} \subseteq \mathbb{R}^N$ is whole $\mathbb{R}^N$ (for any $x \in \overline{\mathcal{R}}$).

We name feasible min-representations of m satisfying the above condition *regular*. One can show that a min-representation $\mathcal{R} \subseteq \mathbb{R}^N$ of an exact game m is regular iff $\mathcal{R} \subseteq \text{ext}\, C(m)$.

Regularity of a min-representation means it consists of the *vertices* of the core.

Tabular arrangement (of the standard min-representation)

Given $m \in \mathsf{E}_\ell(N)$, one can arrange the list of vertices of its core $C(m)$ in the form of a real *array* $x \in \mathbb{R}^{\Upsilon \times N}$, with an auxiliary row-index set Υ:

$$x \;:=\; [\, x(\tau, i)\,]_{\tau\in\Upsilon, i\in N} \quad \text{where} \;\; \mathrm{ext}\, C(m) = \{\, [x(\tau, i)]_{i\in N} : \; \tau \in \Upsilon \,\}.$$

Entries in such a table are non-negative, zero is contained in each column and m is implicitly encoded there because $m(S) = \min_{\tau\in\Upsilon} \sum_{i\in S} x(\tau, i)$ for any $S \subseteq N$.

In this context, the tightness classes correspond to rows:

$$\mathcal{T}_\tau \;:=\; \{\, S \subseteq N \,:\; m(S) = \sum_{i\in S} x(\tau, i)\,\} \qquad \text{for any } \tau \in \Upsilon.$$

The tightness classes are also implicitly encoded in the table.

Regular min-representations of m correspond to *row sub-tables* of the table determined by such $\Gamma \subseteq \Upsilon$ that

$$m(S) = \min_{\tau\in\Gamma} \sum_{i\in S} x(\tau, i) \quad \text{for any } S \subseteq N\,.$$

The linear equation systems

Every regular min-representation $\Gamma \subseteq \Upsilon$ defines a system of linear constraints for the entries of the respective row sub-table:

$$y_\Gamma \;=\; [\, y(\tau,i)\,]_{\tau\in\Gamma, i\in N} \in \mathbb{R}^{\Gamma\times N}\,.$$

(a) $\forall\, \tau\in\Gamma \;\; \forall\, i\in N$ with $\{i\}\in\mathcal{T}_\tau \qquad y(\tau,i)=0,$

(b) $\forall\, S\subseteq N,\ |S|\geq 2,\ \ \forall\, \tau,\rho\in\Gamma$ with $S\in\mathcal{T}_\tau\cap\mathcal{T}_\rho$

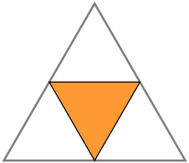
$$\textstyle\sum_{i\in S} y(\tau,i)=\sum_{i\in S} y(\rho,i).$$

Note that the entries of $x_\Gamma\in\mathbb{R}^{\Gamma\times N}$ from the original table satisfy the constraints.

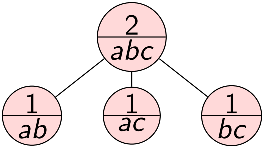
Definition (dimension of a min-representation)

The *dimension* $\dim(\Gamma)$ *of a regular min-representation* specified by $\Gamma\subseteq\Upsilon$ is the dimension (= nullity) of the space of solutions to (a)-(b).

Example: solving the equation system



$$x = \begin{array}{c} \\ \pi \\ \sigma \\ \tau \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{array}$$



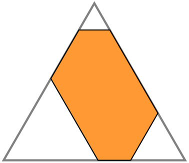
The tightness classes

$\mathcal{T}_\pi = \{\ \emptyset,\ \{a\},\ \{a,b\},\ \{a,c\},\ N\ \}$
$\mathcal{T}_\sigma = \{\ \emptyset,\ \{b\},\ \{a,b\},\ \{b,c\},\ N\ \}$
$\mathcal{T}_\tau = \{\ \emptyset,\ \{c\},\ \{a,c\},\ \{b,c\},\ N\ \}$

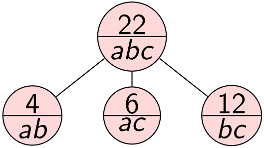
$$y(\pi,a) = 0 \qquad y(\pi,a) + y(\pi,b) = y(\sigma,a) + y(\sigma,b)$$
$$y(\sigma,b) = 0 \qquad y(\pi,a) + y(\pi,c) = y(\tau,a) + y(\tau,c)$$
$$y(\tau,c) = 0 \qquad y(\sigma,b) + y(\sigma,c) = y(\tau,b) + y(\tau,c)$$

$$\sum_{i \in N} y(\pi,i) = \sum_{i \in N} y(\sigma,i) = \sum_{i \in N} y(\tau,i)$$

Example: dimension can be more than one



$$x = \begin{pmatrix} 0 & 4 & 18 \\ 4 & 0 & 18 \\ 0 & 16 & 6 \\ 6 & 16 & 0 \\ 10 & 0 & 12 \\ 10 & 12 & 0 \end{pmatrix}$$



Counterexample:

$$\begin{pmatrix} 0 & 0 & 22 \\ 0 & 0 & 22 \\ 0 & 16 & 6 \\ 6 & 16 & 0 \\ 10 & 0 & 12 \\ 10 & 12 & 0 \end{pmatrix} \neq \alpha \cdot x$$

The main results

Let $m \in \mathsf{E}_\ell(N)$ be a *non-zero* ℓ-standardized exact game.

Proposition (necessity)

If m is extreme in the cone $\mathsf{E}_\ell(N)$ then
the dimension of every its regular min-representation $\Gamma \subseteq \Upsilon$ is 1.

We also showed that, in general, $\Gamma \subseteq \Gamma' \subseteq \Upsilon \;\Rightarrow\; \dim(\Gamma) \geq \dim(\Gamma')$.
In particular, it is enough to test "minimal" regular min-representations.

Proposition (sufficiency)

If there exists the least regular min-representation of m then m is extreme iff the dimension of the least regular min-representation $\Gamma^\star \subseteq \Upsilon$ is 1.

Note that the oxytrophy condition $\Gamma^\star = \Upsilon$ discussed by Rosenmüller (2000) implies trivially the existence of the least regular min-representation.

Conclusions: web platforms

We have prepared a web platform for testing the extremity of an ℓ-standardized integer-valued exact game, available at

http://gogo.utia.cas.cz:3838/exact-and-supermodular/ .

It also allows one to test the extremity in a smaller supermodular cone. Moreover, it allows one to test whether an exact game is oxytropic.

We have tested our criterion on 41 permutation types of 398 extreme ℓ-standardized exact games over 4 variables.
In 39 types of these the least regular min-representation exists.

We conjectured that a necessary and sufficient condition for the extremity is that the dimension of every "minimal" regular min-representation is 1.

Nonetheless, meanwhile we found (slightly different) necessary and sufficient condition in terms of certain *"finest" feasible min-representation*.
The poster presentation contains some details about that fresh result and (another) web platform based on that more general approach.