Efficient algorithms for checking avoiding sure loss

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Outline

- Avoiding sure loss
- Results and contributions
  - Linear programs
  - Size reduction
  - Methods for solving linear programs
  - Degeneracy and stalling
  - Early stopping
  - Closed form for initial feasible points
- Improve three methods: simplex, affine scaling and primal-dual
- Benchmarking

Desirability axioms

- A possibility space Ω
- A gamble $f : \Omega \to \mathbb{R}$

Q: How should we reason with desirable gambles?

Suppose we are offered:

Outcomes	A	B	C
f_1	-5	-1	-2
f_2	30	20	0
f_3	-1	2	-1
f_4	-50	100	-50
$f_2 + f_4$	-20	120	-50

Desirability axioms

(D1) Do not accept sure loss.

(D2) Accept sure gain.

(D3) Accept positive scaled invariance.

(D4) Accept positive combination of desirable gambles. [2]

Avoiding sure loss

Definition 1

A set of desirable gambles $\mathcal{D}$ is said to *avoid sure loss* if for all $n \in \mathbb{N}$, $\lambda_1, \cdots, \lambda_n \geq 0$ and $f_1, \cdots, f_n \in \mathcal{D}$ [5]:

$$\sup_{\omega \in \Omega} \left(\sum_{i=1}^{n} \lambda_i f_i(\omega) \right) \geq 0. \tag{1}$$

Linear programs

Theorem 2

A set of desirable gambles $\mathcal{D}$ avoids sure loss if and only if the optimal value of (P1) is zero, or if the dual problem has feasible solutions [7].

$$
\begin{aligned}
\text{(P1)} \quad & \min \quad \alpha \\
\text{subject to} \quad & \forall \omega \in \Omega : \sum_{i=1}^{n} \lambda_i f_i(\omega) - \alpha \leq 0 \\
\text{where} \quad & \lambda_i \geq 0 \quad (\alpha \text{ free}).
\end{aligned}
$$

$$
\begin{aligned}
\text{(D1)} \quad & \max \quad 0 \\
\text{subject to} \quad & \forall f_i : \sum_{\omega \in \Omega} f_i(\omega) p(\omega) \geq 0 \\
& \sum_{\omega \in \Omega} p(\omega) = 1 \\
\text{where} \quad & p(\omega) \geq 0.
\end{aligned}
$$

Size reduction

An alternative linear program is slightly smaller in size and has only non-negative variables:

Theorem 3

Choose any $\omega^* \in \Omega$. A set $\mathcal{D}$ avoids sure loss if and only if the optimal value of (P2) is zero, or if (D2) has feasible solutions [3].

$$\text{(P2)} \quad \min \sum_{i=1}^{n} \lambda_i f_i(\omega^*) + \alpha$$

$$\text{subject to} \quad \forall \omega \neq \omega^* : \sum_{i=1}^{n} \lambda_i (f_i(\omega^*) - f_i(\omega)) + \alpha \geq 0$$

$$\text{where} \quad \lambda_i, \alpha \geq 0.$$

$$\text{(D2)} \quad \max \quad 0$$

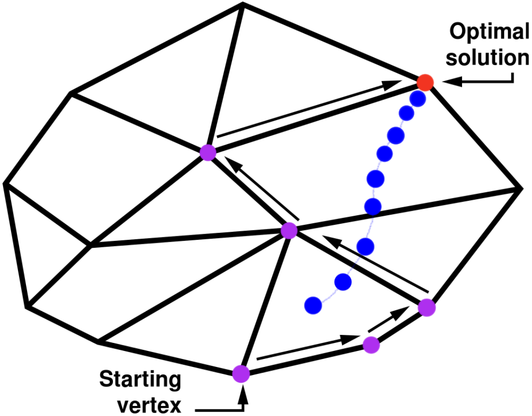
$$\text{subject to } \forall f_i \in \mathcal{D} : \sum_{\omega \neq \omega^*} (f_i(\omega^*) - f_i(\omega)) p(\omega) \leq f_i(\omega^*)$$

$$\sum_{\omega \neq \omega^*} p(\omega) \leq 1$$

$$\text{where} \quad p(\omega) \geq 0.$$

Algorithms for solving linear programs

- Simplex
- Affine scaling
- Primal-dual



Degeneracy

- Simplex
  - Cycling: infinite iterations and no convergence.
  - Stalling: finite iterations, but in exponential time [1].
- Affine scaling
  - A restriction on a step-size [4].
- Primal-dual
  - Affecting numerical performance [1].

Early stopping

Lemma 4

(Adapted from [6]) The linear programming problem

$$\begin{aligned} \min \quad & c^\intercal x && (2) \\ \textit{subject to} \quad & Ax \geq 0 && (3) \end{aligned}$$

either has an optimal value that is zero, or is unbounded.

- Can stop when a current value is negative.
- Extra stopping for affine scaling and primal-dual.

Starting points

- To start those methods, we need an initial point.
- Simplex and Affine scaling
  - Closed form for both primal and dual problems.
- Primal-dual
  - Closed form for primal problem.

Overall comparison among improved methods

Comparison	simplex	affine scaling	primal-dual
Stalling	$-$	$+$	$+$
Stop early	$-$	$+$	$+$
Starting points	$+$	$+$	$\pm$
Convergence speed	$-$	$+$	$++$
Complexity per step	$++$	$-$	$-$

Conclusion

- Simplex is not normally efficient for checking avoiding sure loss due to degeneracy.
- Because of early stopping rules, affine scaling and primal-dual are much more efficient, especially when sets of desirable gambles do not avoid sure loss.
- Overall performance for checking avoiding sure loss:
  primal-dual $>$ affine scaling $>$ simplex.

References

[1] Christodoulos A. Floudas and Panos M. Pardalos, editors. *Encyclopedia of Optimization, Second Edition*. Springer, 2009.

[2] Enrique Miranda and Gert de Cooman. *Introduction to Imprecise Probabilities*, chapter Lower prevision, pages 28–55. Wiley, 2014.

[3] Nawapon Nakharutai. Computing with lower previsions using linear programming. Master's thesis, Durham University, 2015. unpublished thesis.

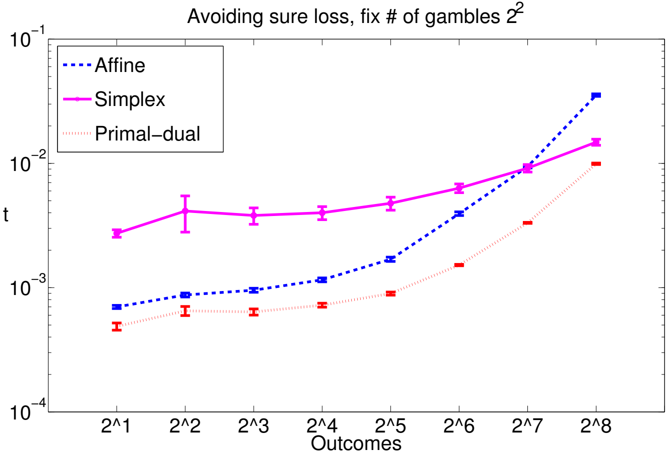
[4] Romesh Saigal. *Linear programming : a modem integrated analysis*. Springer Science+Business Media New York, 1995.

[5] Matthias C. M. Troffaes and Gert de Cooman. *Lower Previsions*. Wiley Series in Probability and Statistics. Wiley, 2014.

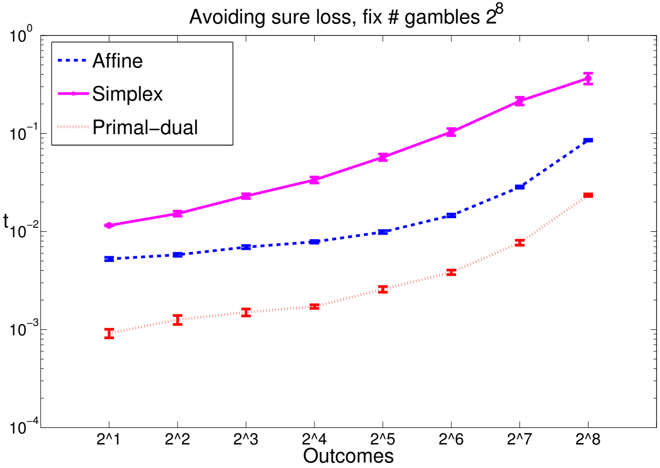
[6] Robert J. Vanderbei. *Linear Programming: Foundations and Extensions, Second edition*. Springer, 2001.

[7] Peter Walley. *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, London, 1991.

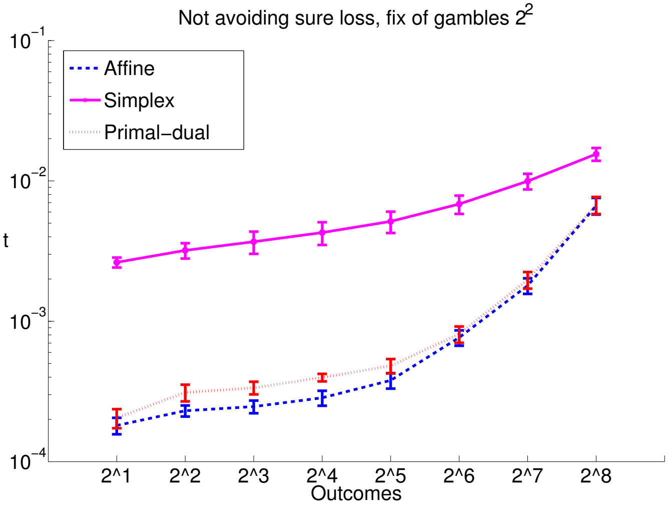
Benchmarking improved methods



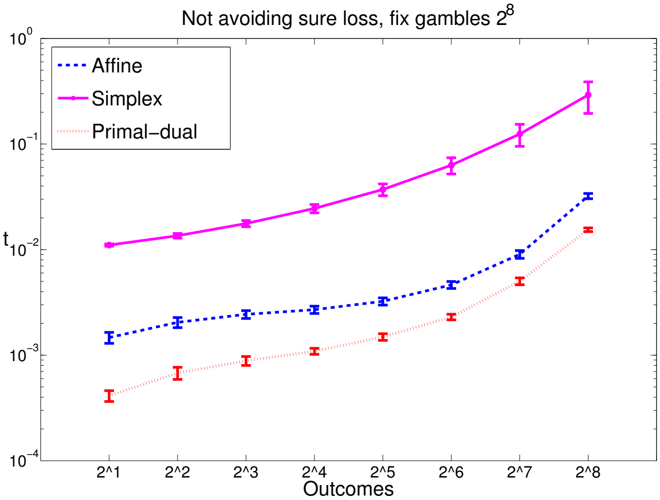
Benchmarking improved methods



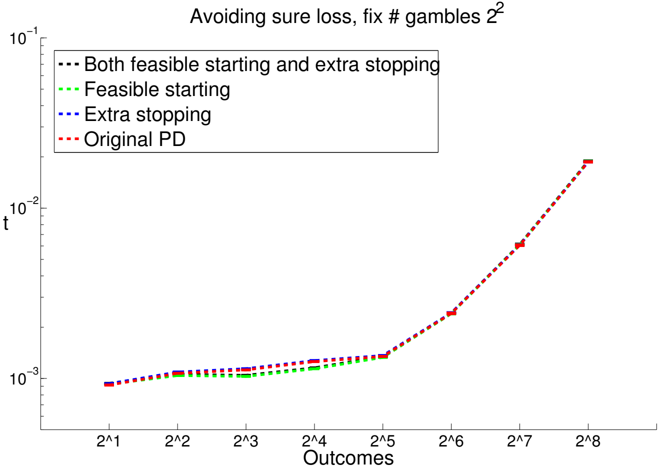
Benchmarking improved methods



Benchmarking improved methods



Different improved primal-dual methods



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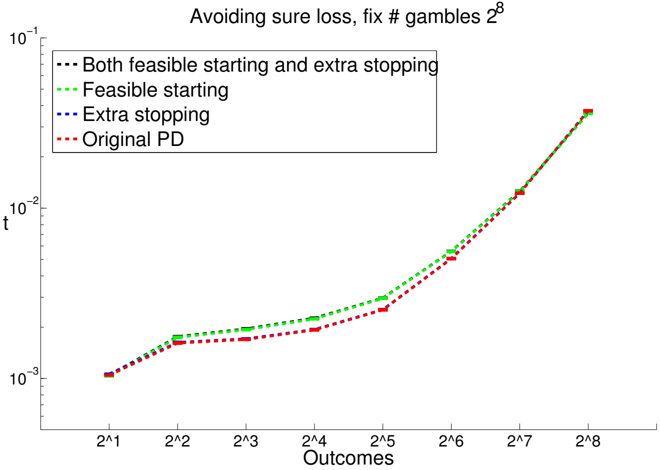
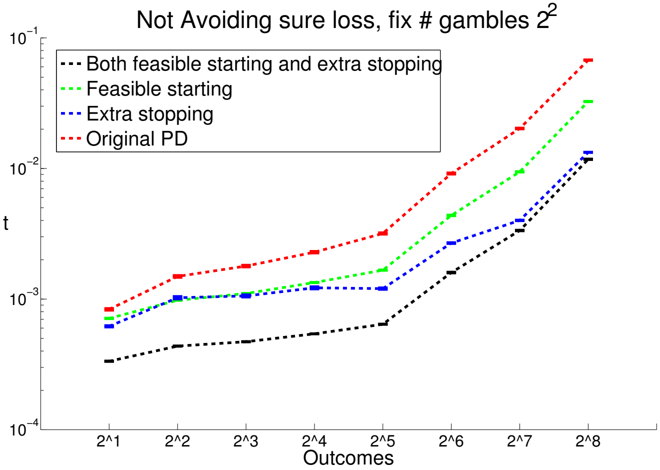


Different improved primal-dual methods



Different improved primal-dual methods