Implementation of linear core-based criterion for testing extreme exact games

Václav Kratochvíl, Milan Studený Institute of Information Theory and Automation of the CAS, Prague

velorex@utia.cas.cz, studeny@utia.cas.cz

Notation	Extreme exact game - A		Non-extreme exact game - B	
Let N be a finite non-empty set of variables, $ N \ge 2$, $\mathcal{P}(N) := \{S : S \subseteq N\}$. \mathbb{R}^N will denote the set of real vectors whose components are indexed by elements of N. A set function m is l -standardized if $\forall S \le 1 : m(S) = 0$.	\emptyset a b c ab ac bc abc $0 \ 0 \ 0 \ 0 \ \frac{1}{2} \ \frac{1}{2} \ \frac{1}{2} \ 1$		\emptyset a b c ab ac bc abc $0 \ 0 \ 0 \ 0 \ 2/11 \ 3/11 \ 6/11 \ 1$	
Exact games & lower probabilities				
Let $m : \mathcal{P}(N) \to \mathbb{R}, m(\emptyset) = 0$ be a game. Its <i>core</i> is a polytope $C(m)$ in \mathbb{R}^N defined by	$\frac{a \ b \ c}{\frac{1/2}{0} \ \frac{1/2}{0}}$	<u>abc</u> 101	$ \begin{array}{ccccccccccccccccccccccccccccccccccc$	$\frac{a b c}{0 2 9}$
$\{x \in \mathbb{R}^N : \sum_{i \in N} x_i = m(N) \& \forall S \subseteq N \sum_{i \in S} x_i \ge m(S)\}.$	$\frac{1}{2} \frac{1}{2} 0$	$\begin{array}{c}1 1 0\\0 1 1\end{array}$	$\begin{array}{cccc} 0 & 8/_{11} & 3/_{11} \\ 5/_{11} & 6/_{11} & 0 \end{array}$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$

If $C(m) \neq \emptyset$ then we say the game is exact if

$$\forall S \subseteq N \; \exists x \in C(m) \; \sum_{i \in S} x_i = m(S).$$

Non-negative exact game normalized by m(N) = 1 is a coherent lower probability.

Polyhedral cone

The collection of exact games is a rational polyhedral cone. Moreover, *l*-standardized exact games form a pointed rational cone $E_l(N)$. Every pointed polyhedral cone has finitely many extreme rays.

Extreme exact game

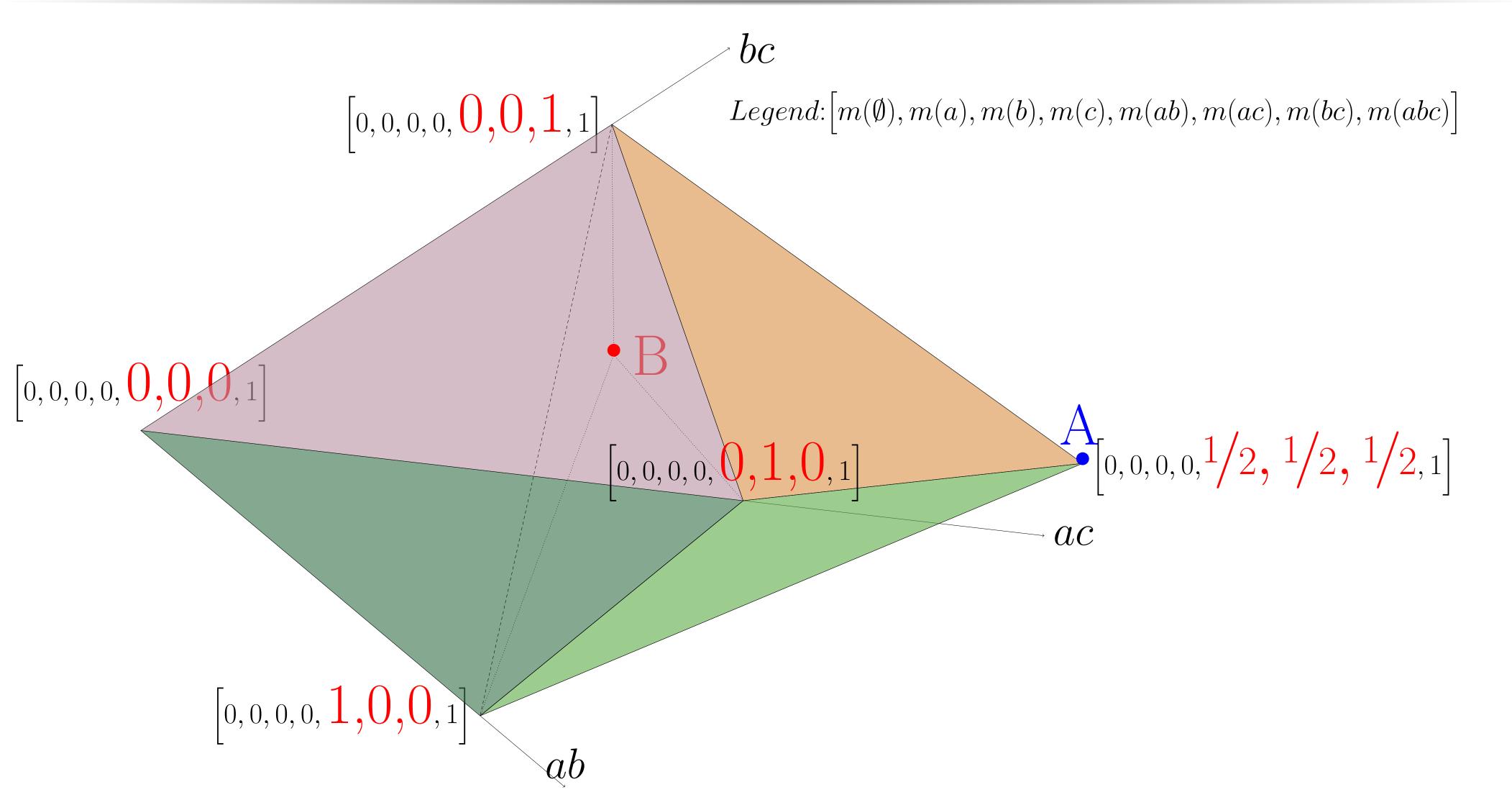
An *l*-standardized exact game is called *extreme* if it generates an extreme ray of $E_l(N)$. Since $E_l(N)$ is a rational cone, every extreme game is a multiple of an integer-valued game $m : \mathcal{P}(N) \to \mathbb{R}$. We can limit ourselves to **integer-valued games**. This is important for the implementation and we offer a criterion testable on a computer.

Feasible min-representations

A min-representation of a game m is a finite set $\mathcal{R} \subseteq \mathbb{R}^N$ such that $m(S) = \min_{x \in \mathcal{R}} \sum_{i \in S} x_i$ for any $S \subseteq N$.

$0 - \frac{1}{2} - \frac{1}{2}$		3/11 8/11 0	3 8 0
		2/11 0 9/11	$2\ 0\ 9$
		5/11 0 6/11	$5 \ 0 \ 6$
			1.

Polyhedral cone - 3D visualization on selected coordinates



A game is exact \Leftrightarrow it has a feasible min-representation, that is, $\sum_{i \in N} x_i = m(N)$ holds for any $x \in \mathcal{R}$.

Standard min-representation

An exceptional standard min-representation $\overline{\mathcal{R}}$ of an exact game *m* consists of (the set of) vertices of the core C(m).

Tightness structure

Let $\mathcal{R} \subseteq \mathbb{R}^N$ be a feasible min-representation of an exact game. For every vector $x \in \mathcal{R}$ a collection of sets

 $\mathcal{T}_x^m = \{ S \subseteq N : \sum_{i \in S} x_i = m(S) \}$

is called its **tightness class**. List of all tightness classes for all vectors from \mathcal{R} is called **tightness structure** $\mathcal{T}^{\mathcal{R}}$. Given two feasible min-representations \mathcal{R}, \mathcal{L} of m we say that the $\mathcal{T}^{\mathcal{R}}$ refines $\mathcal{T}^{\mathcal{L}}$ if

 $\forall x \in \mathcal{R} \; \exists y \in \mathcal{L} : \; \mathcal{T}_x^{\mathcal{R}} \subseteq \mathcal{T}_y^{\mathcal{L}}.$

The standard min-representation has the coarsest tightness structure.

Finest min-representation

There exist the finest min-representations = representations with the finest tightness structure. The finest min-representations are

N = 4 example

Extreme exact game	Tightness structure(s)			
$\ensuremath{\emptyset}$ a b c d ab ac ad bc bd cd abc abd acd bcd abcd		abcd		
0 0 0 0 0 2 1 0 1 0 0 2 2 1 1 3	$\begin{array}{c c c c c c c c c c c c c c c c c c c $	•		
Reduced dimension	$\begin{array}{c c c c c c c c c c c c c c c c c c c $	•		
Reduced dimension of the finest min-representation	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	•		
equals to 1. The game is extreme.	$\frac{(\frac{3}{2},\frac{3}{2},0,0)}{(\frac{3}{2},\frac{3}{2},0,0)} \bullet \bullet \bullet$	•		

Software - http://gogo.utia.cas.cz/finest-min-representation/

Gogo.utia.cas.cz:3838/finest-min-representation/	EI 90% C	Q Hledat	☆ 自 🔸 🎓 💟 🕒 Ξ

Criterion to Recognize Extreme Exact Games

Number of variables (= players)
Choose that by drawing the circle:
2 4

Input array

You can enter the values of an input min-representation using the array Input/saving array on the right-hand side. The values should be non-negative integers and the sum in any row should be the same for different rows. Moreover, to ensure the resulting represented exact game to be

not unique but the corresponding tightness structure is unique. We have an algorithm to obtain it!

Reduced dimension

For each *finest min-representation* we can compute the so-called **reduced dimension** (technical details omitted). Let \mathcal{R}, \mathcal{L} be two feasible min-representation such that $\mathcal{T}^{\mathcal{R}}$ refines $\mathcal{T}^{\mathcal{L}}$. Then

 $1 \leq \operatorname{redim}(\mathcal{L}) \leq \operatorname{redim}(\mathcal{R}) \leq 2^{|N|} - |N| - 1.$

Generally, reduced dimension is the dimension of the smallest face of the pointed polyhedral cone containing respective game.

Criterion

An l-standardized game is extreme if the reduced dimension of its finest min-representation is 1.

This research is supported by the grant project GAČR no. 16-12010S.



There are basically two ways to enter the data:

Represented game

all sets are zeros.

 either you can enter the values of the game in the frame entitled Represented game,
 or you can enter a min-representation in the table entitled Input/saving array.

Here you can enter the values of the game. These should be

automatically set to zero and the initial pre-defined values for

The standard min-representation of the game consisting of the

list of all vertices of the core will be automatically computed

A m

using R package rcdd. It will appear in the array entitled

Core/standard min-representation.

a 0 ab 2 abc 2 abcd 3

b 0 ac 1 abd 2

c 0 ad 0 acd 1

d 0 bc 1 bcd 1

bd 0

cd 0

Clear the game

non-negative integers. The value for the empty set is

standardized, every column of the input min-representation should contain at least once zero. Additional rows for the input can be generated using the button called *Add row*.

The initial pre-defined values in every newly generated row are zeros. After entering the input min-representation click on the button with the leftdirected arrow. The represented game will be computed and will appear in the frame **Represented game**. Moreover, the vertices of the core of the represented game will be automatically computed and will appear in the array entitled **Core/standard min-representation**. Note that this standard min-representation of the game can have a different number of rows than the input min-representation. Using the button with the rightdirected arrow you can save it in the (originally input) array on the right-hand side.

Core/standard min-representation

 b
 c
 d

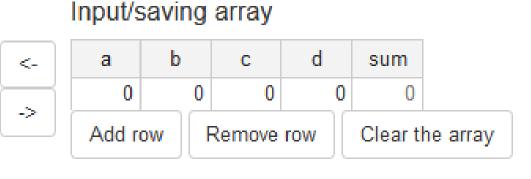
 2
 1
 0

 0
 1
 0

 2
 0
 0

 1
 0
 0

 1
 0
 1



A test based on finest min-representation

By clicking on the button Reduced dimension you can start a test whether the represented exact game is extreme. The test is based on the finest min-representation. One computes the reduced dimension for the finest min-representation and the game is extreme if and only if the reduced dimension is 1.

Finest min-representation

	а	b	С	d	Reduced dimension	Reduced dimension of the Finest min-representation is 1 (constraint matrix nullity is 2).
i	0.00	2.00	1.00	0.00	L	
	2.00	0.00	1.00	0.00		
	1.50	1.50	0.00	0.00		
	1.00	1.00	0.00	1.00		